Ast 321:

CMB Polarization, Large Scale Structure, Dark Energy

Wayne Hu

Polarization Anisotropy

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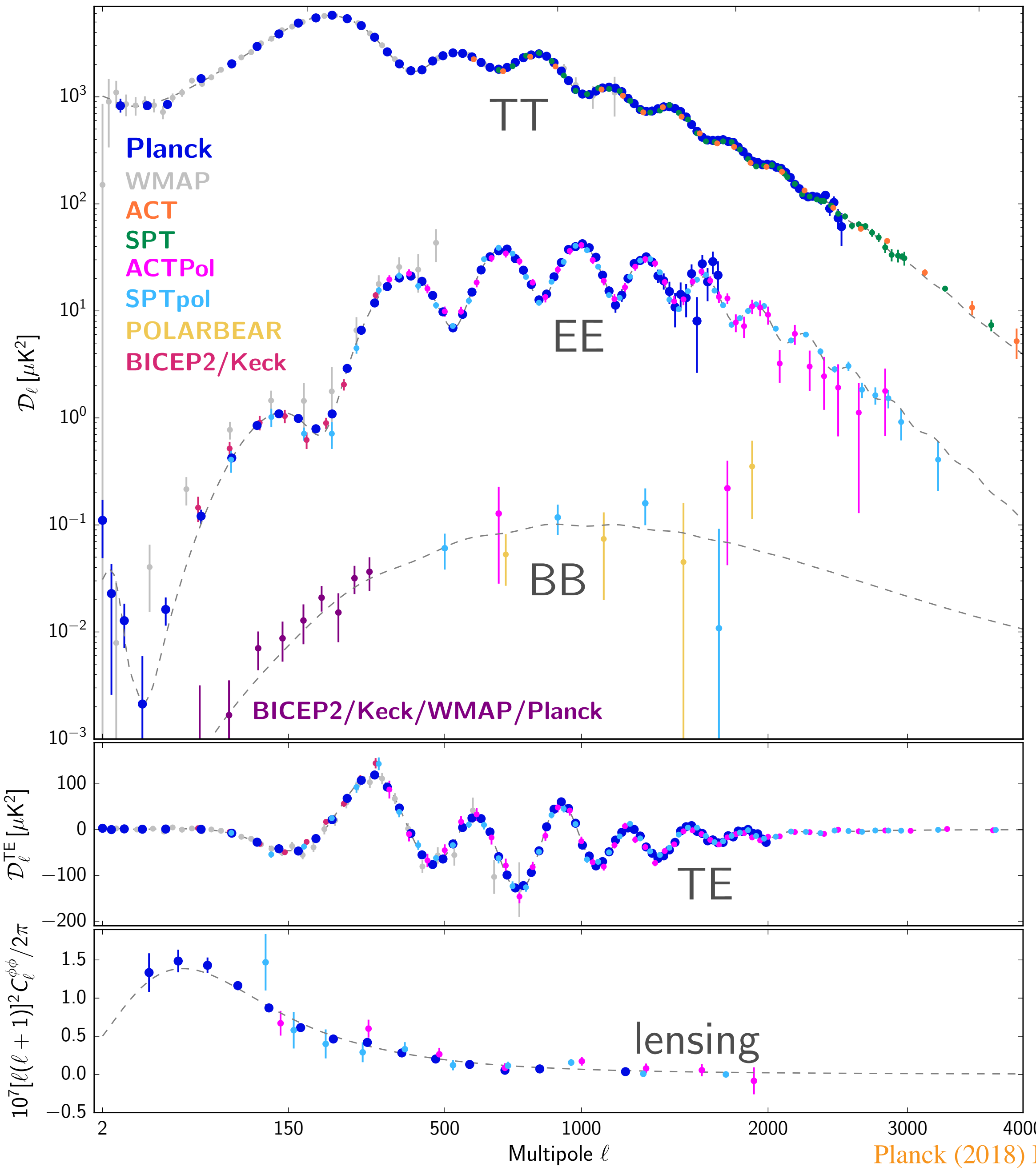
CMB Power Spectra

- Power spectra of CMB
  - temperature
  - polarization
  - lensing



Planck (2018) I

Linear Polarization: Stokes Q, U

- $Q \propto \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle$, $U \propto \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle$.
- Counterclockwise rotation of axes by $\theta = 45°$

$$E_1 = (E_1' - E_2')/\sqrt{2}\,, \quad E_2 = (E_1' + E_2')/\sqrt{2}$$

- $U \propto \langle E_1' E_1'^* \rangle - \langle E_2' E_2'^* \rangle$, difference of intensities at $45°$ or Q'
- More generally, $\mathbf{P}$ transforms as a tensor under rotations and

$$Q' = \cos(2\theta) Q + \sin(2\theta) U$$
$$U' = -\sin(2\theta) Q + \cos(2\theta) U$$

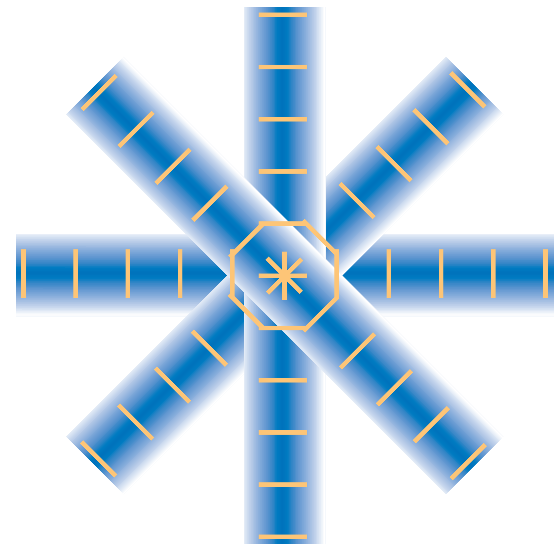
or

$$Q' \pm iU' = e^{\mp 2i\theta}[Q \pm iU]$$

acquires a phase under rotation and is a spin ± 2 object

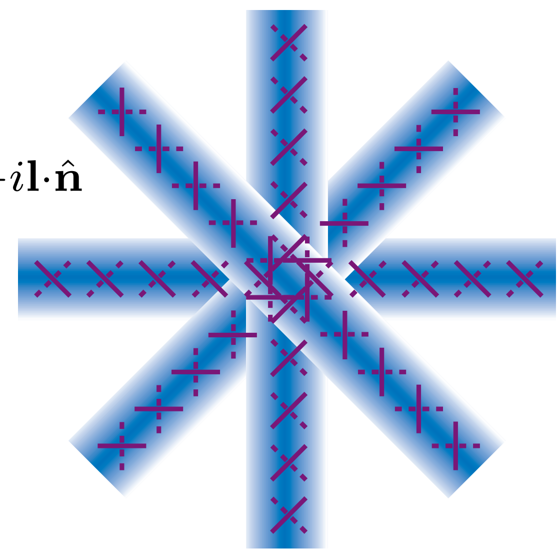
Coordinate Independent Representation

- Two directions: orientation of polarization and change in amplitude, i.e. Q and U in the basis of the Fourier wavevector (pointing with angle ϕ_l) for small sections of sky are called E and B components



$$E(\mathbf{l}) \pm iB(\mathbf{l}) = -\int d\hat{\mathbf{n}}[Q'(\hat{\mathbf{n}}) \pm iU'(\hat{\mathbf{n}})]e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}}$$

$$= -e^{\mp 2i\phi_l}\int d\hat{\mathbf{n}}[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})]e^{-i\mathbf{l}\cdot\hat{\mathbf{n}}}$$



- For the B-mode to not vanish, the polarization must point in a direction not related to the wavevector - not possible for density fluctuations in linear theory

- Generalize to all-sky: eigenmodes of Laplace operator of tensor

Spin Harmonics

- Laplace Eigenfunctions

$$\nabla^2 {}_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1] = -[l(l+1)-4]_{\pm 2}Y_{\ell m}[\boldsymbol{\sigma}_3 \mp i\boldsymbol{\sigma}_1]$$

- Spin s spherical harmonics: orthogonal and complete

$$\int d\hat{\mathbf{n}}\, {}_sY_{\ell m}^*(\hat{\mathbf{n}})\, {}_sY_{\ell' m'}(\hat{\mathbf{n}}) = \delta_{\ell\ell'}\delta_{mm'}$$

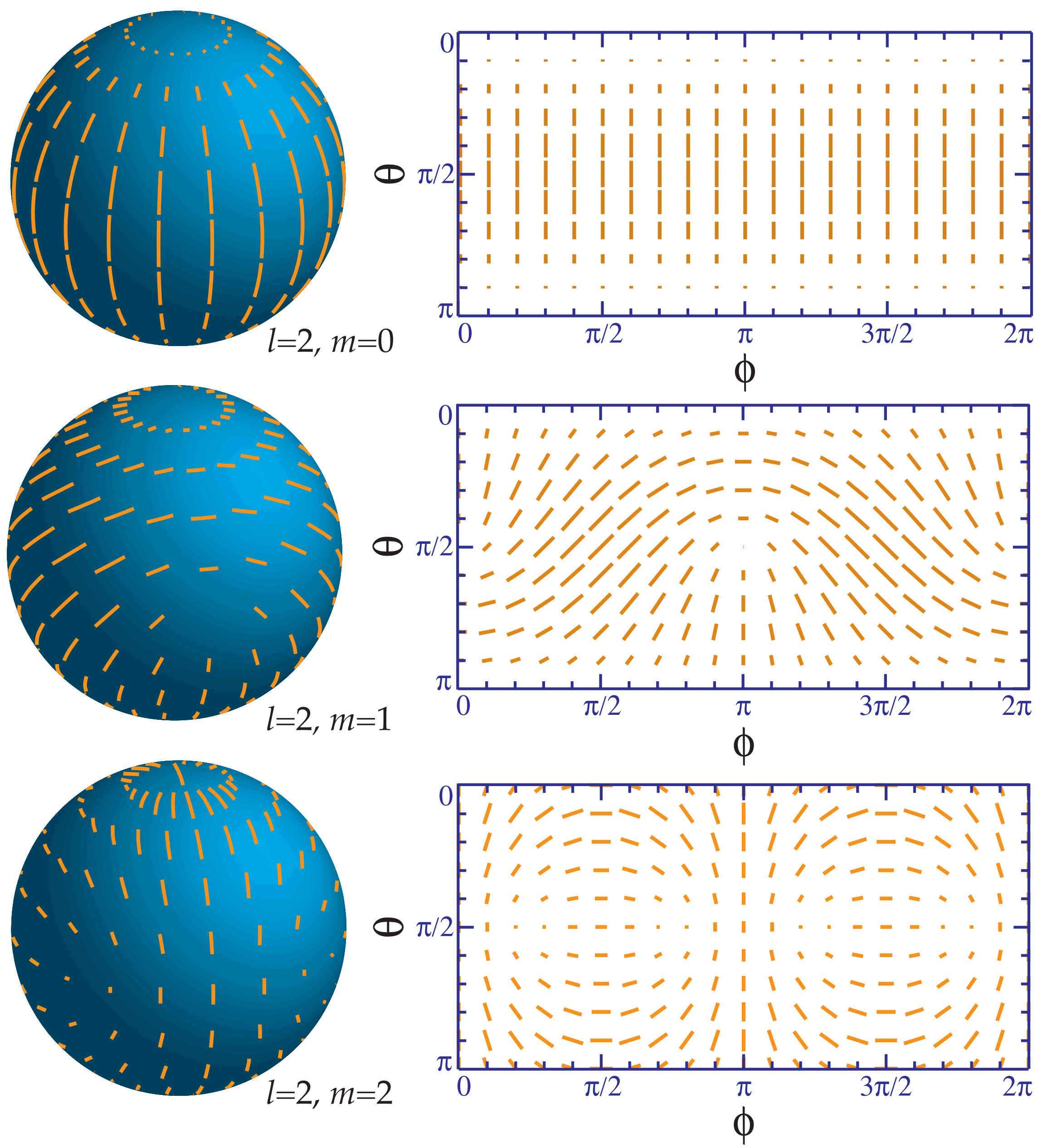
$$\sum_{\ell m} {}_sY_{\ell m}^*(\hat{\mathbf{n}})\, {}_sY_{\ell m}(\hat{\mathbf{n}}') = \delta(\phi - \phi')\delta(\cos\theta - \cos\theta')$$

where the ordinary spherical harmonics are $Y_{\ell m} = {}_0Y_{\ell m}$

- Given in terms of the rotation matrix

$${}_sY_{\ell m}(\beta\alpha) = (-1)^m \sqrt{\frac{2\ell+1}{4\pi}} D^{\ell}_{-ms}(\alpha\beta 0)$$

Polarization Multipoles



Statistical Representation

- All-sky decomposition

$$[Q(\hat{\mathbf{n}}) \pm iU(\hat{\mathbf{n}})] = \sum_{\ell m} [E_{\ell m} \pm iB_{\ell m}]_{\pm 2}Y_{\ell m}(\hat{\mathbf{n}})$$

- Power spectra

$$\langle E_{\ell m}^* E_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell^{EE}$$

$$\langle B_{\ell m}^* B_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell^{BB}$$

- Cross correlation

$$\langle \Theta_{\ell m}^* E_{\ell m} \rangle = \delta_{\ell\ell'} \delta_{mm'} C_\ell^{\Theta E}$$

others vanish if parity is conserved

Thomson Scattering

- Polarization state of radiation in direction $\hat{\mathbf{n}}$ described by the intensity matrix $\langle E_i(\hat{\mathbf{n}})E_j^*(\hat{\mathbf{n}})\rangle$, where $\mathbf{E}$ is the electric field vector and the brackets denote time averaging.
- Differential cross section

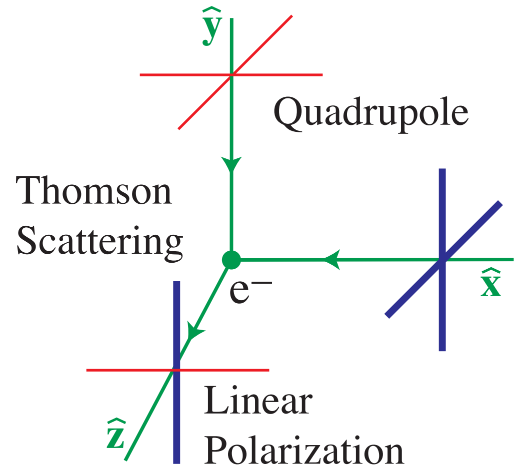
$$\frac{d\sigma}{d\Omega} = \frac{3}{8\pi}|\hat{\mathbf{E}}' \cdot \hat{\mathbf{E}}|^2 \sigma_T\,,$$

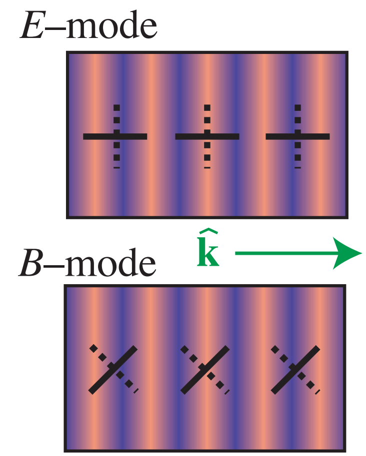
where $\sigma_T = 8\pi\alpha^2/3m_e$ is the Thomson cross section, $\hat{\mathbf{E}}'$ and $\hat{\mathbf{E}}$ denote the incoming and outgoing directions of the electric field or polarization vector.

- Summed over angle and incoming polarization

$$\sum_{i=1,2} \int d\hat{\mathbf{n}}' \frac{d\sigma}{d\Omega} = \sigma_T$$

Polarization Generation

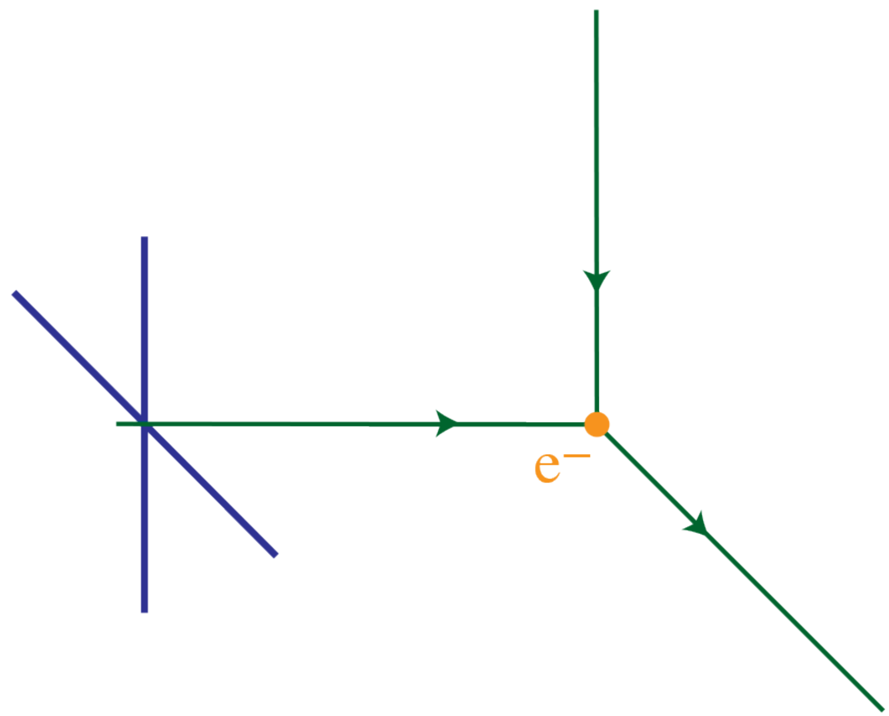




- Heuristic:
  incoming radiation shakes
  an electron in direction
  of electric field vector $\hat{\mathbf{E}}'$
- Radiates photon with
  polarization also in direction $\hat{\mathbf{E}}'$
- But photon cannot be longitudinally polarized so that scattering into 90° can only pass one polarization
- Linearly polarized radiation like polarization by reflection
- Unlike reflection of sunlight, incoming radiation is nearly isotropic
- Missing from direction orthogonal to original incoming direction
- Only quadrupole anisotropy generates polarization by Thomson scattering

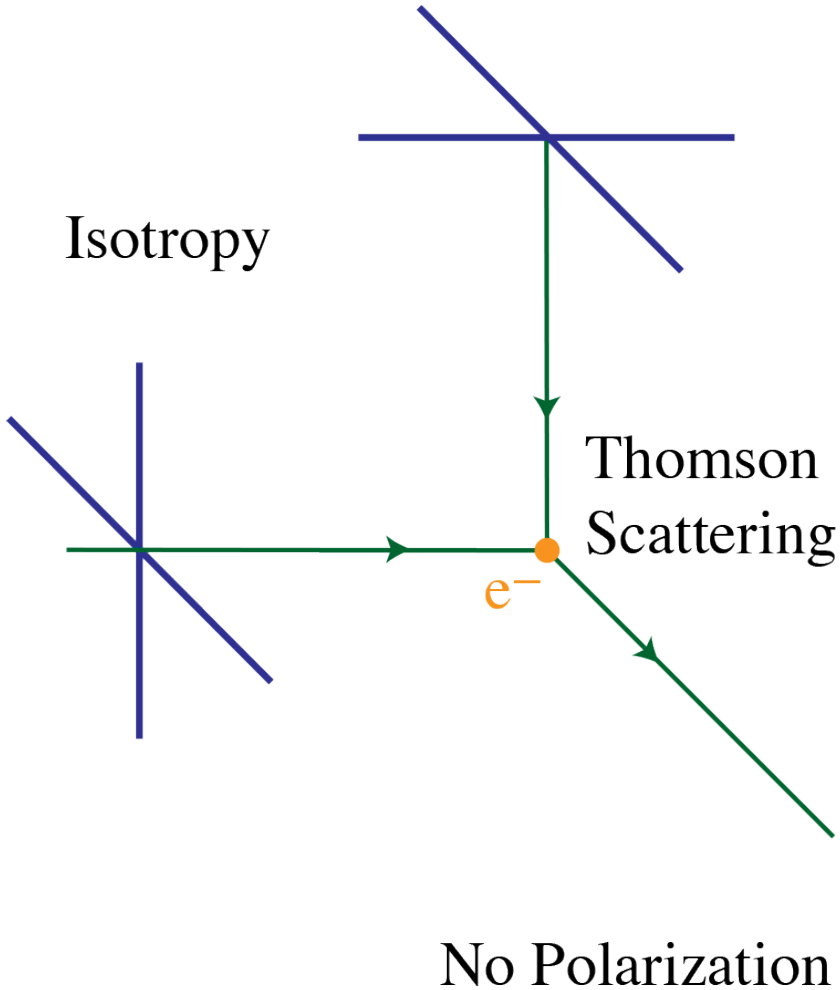
Polarization Generation

- Single direction



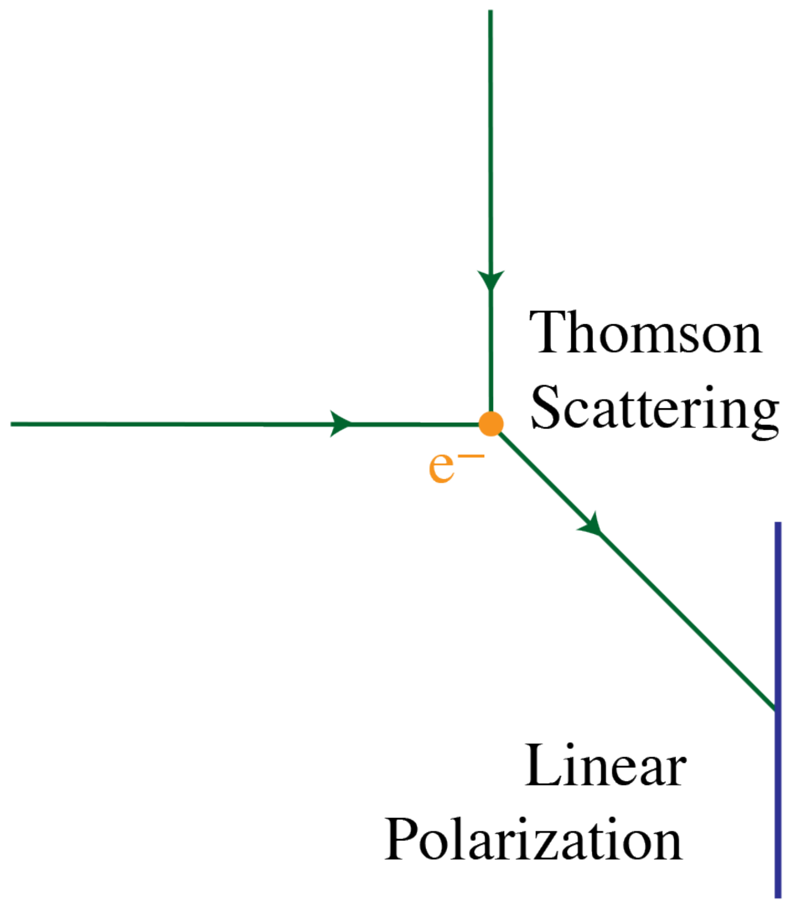
Polarization Generalization

- Isotropic distribution



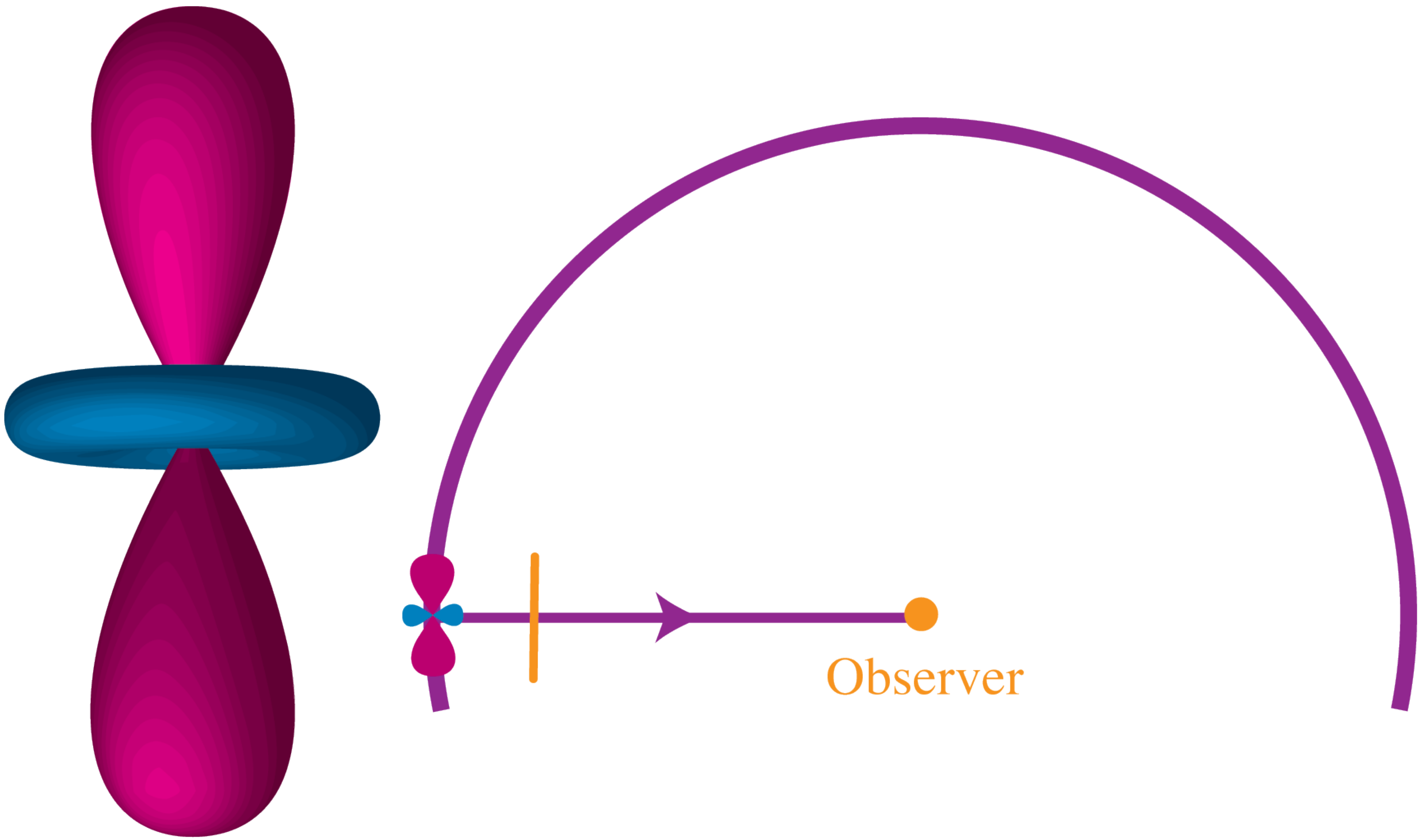
Polarization

- Quadrupole distribution



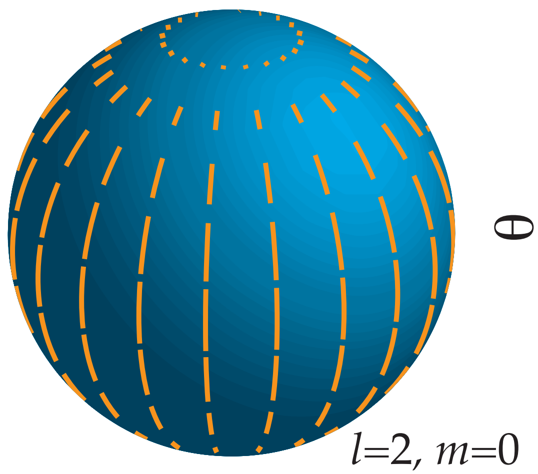
Polarization

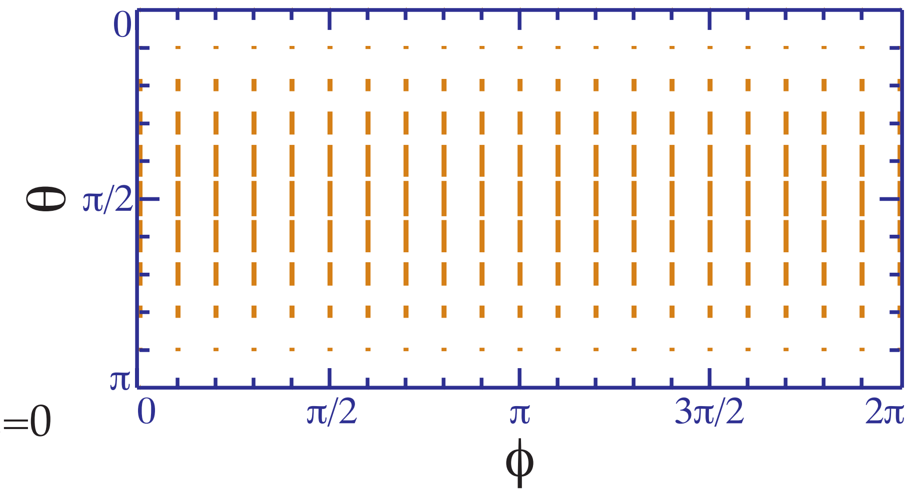
- Scalar quadrupole to E mode

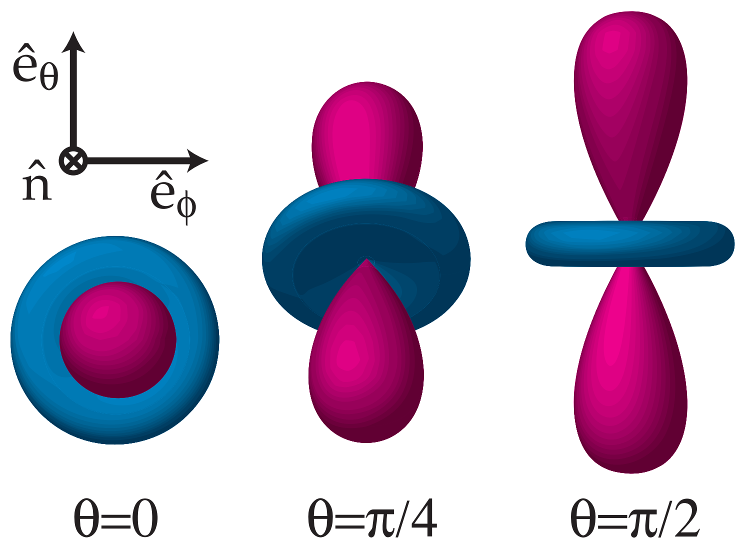


Polarization

- Scalars: $\ell = 2, m = 0$ E mode pattern - quadrupole viewing angle



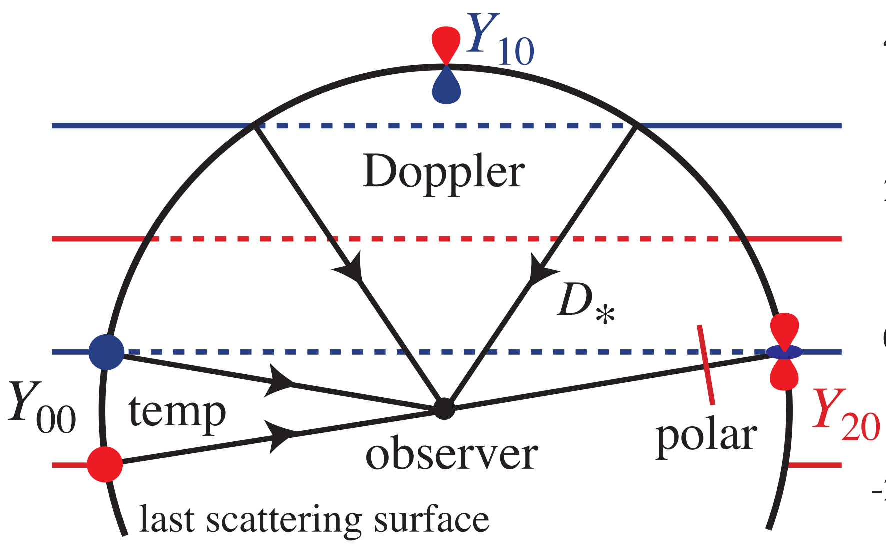


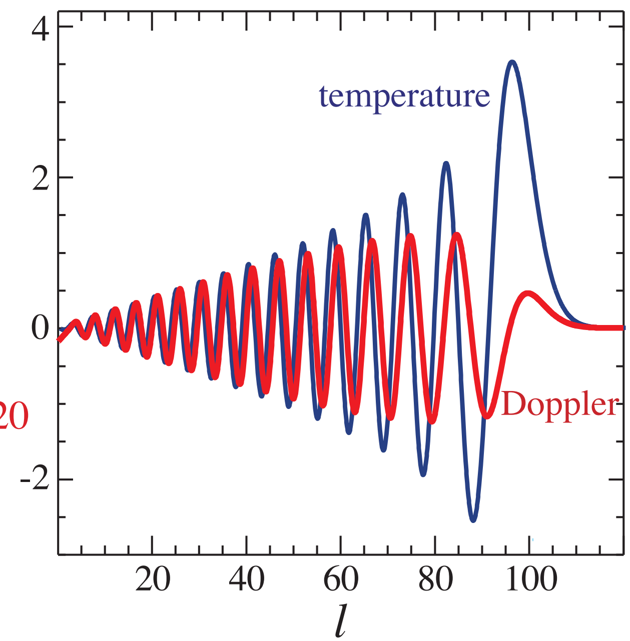


Polarized Radiative Transfer

- Source of linear polarization is the radiation quadrupole
- Quadrupolar structure transferred through plane wave or orbital angular momentum onto polarization anisotropy
- Recall monopole and dipole emission structure – same procedure except couple to $s = \pm 2, \ell = 2$:

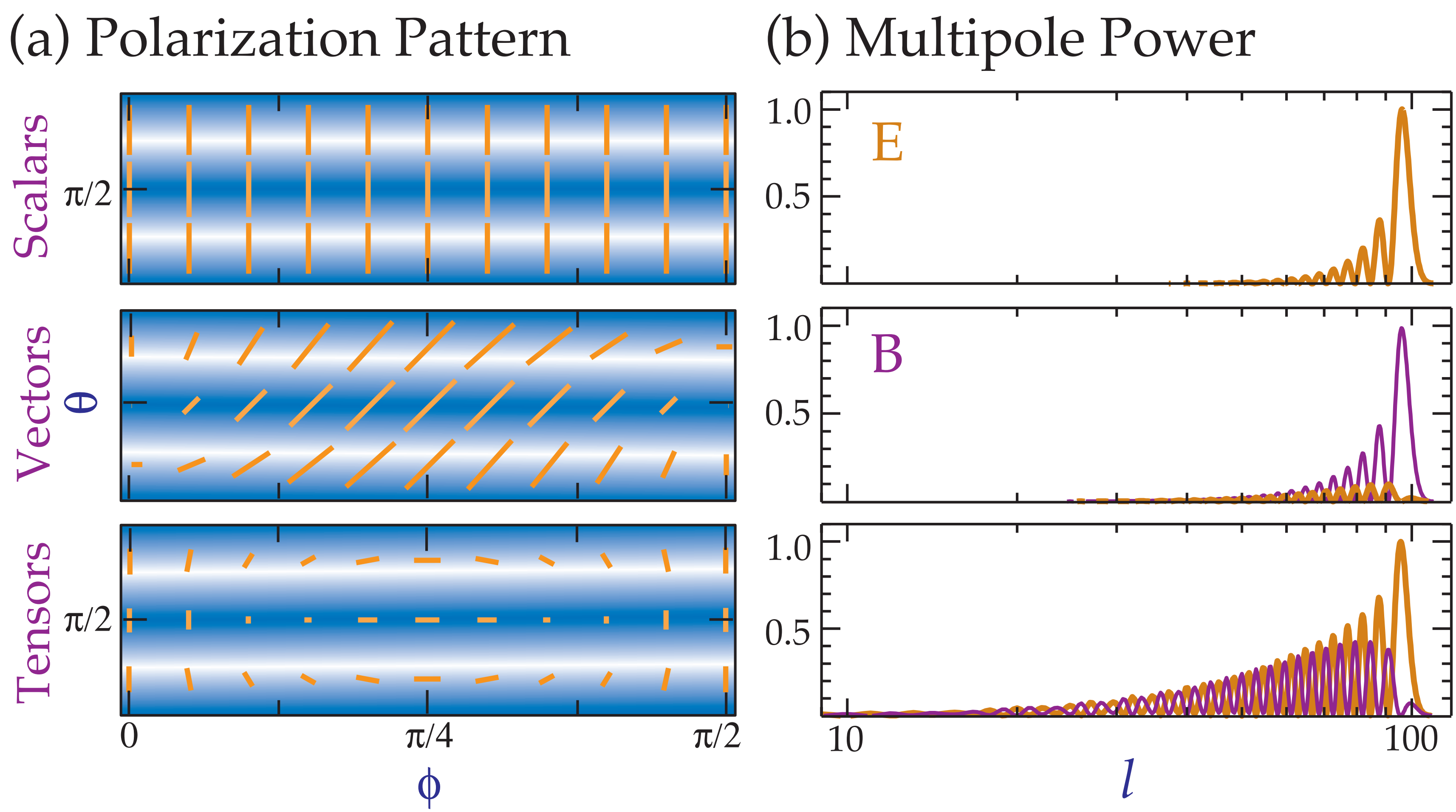
$$_{\pm 2}Y_{2m} Y_{\ell 0} \rightarrow {}_{\pm 2}Y_{(\ell-2)m} \cdots {}_{\pm 2}Y_{(\ell+2)m}$$





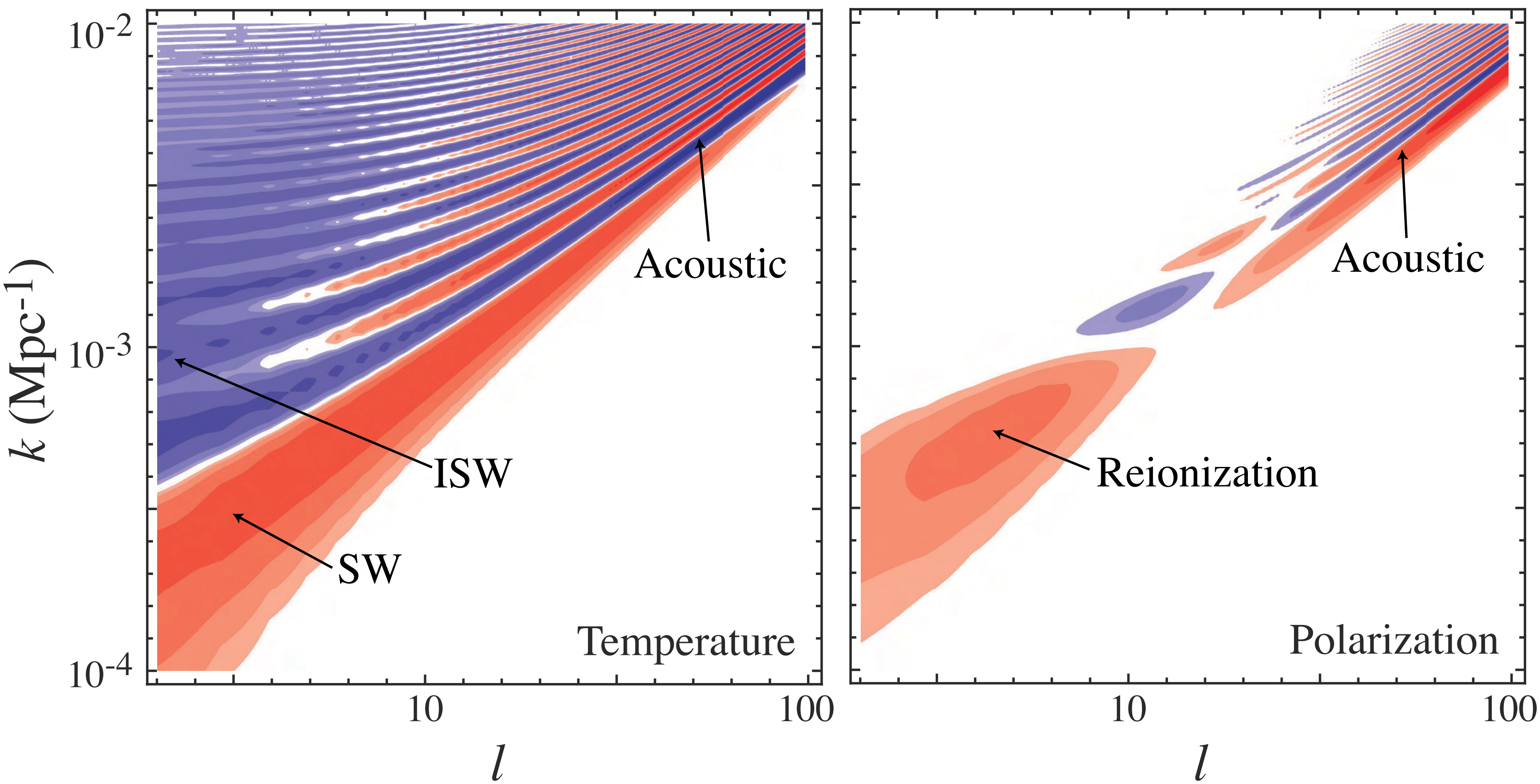
Polarization Transfer

- A polarization source function with $\ell = 2$, modulated with plane wave orbital angular momentum
- Scalars have no B mode contribution, vectors mostly B and tensor comparable B and E



Radiation Transfer Function

- Radiation transfer function takes initial curvature inhomogeneity in k to anisotropy in ℓ



Acoustic Polarization

- Break down of tight-coupling leads to quadrupole anisotropy of

$$\pi_\gamma \approx \frac{k}{\dot\tau} v_\gamma$$

- Scaling $k_D = (\dot\tau/\eta_*)^{1/2} \to \dot\tau = k_D^2 \eta_*$
- Know: $k_D s_* \approx k_D \eta_* \approx 10$
- So:

$$\pi_\gamma \approx \frac{k}{k_D} \frac{1}{10} v_\gamma$$

$$\Delta_P \approx \frac{\ell}{\ell_D} \frac{1}{10} \Delta_T$$

Acoustic Polarization

- Gradient of velocity is along direction of wavevector, so polarization is pure E-mode
- Velocity is 90° out of phase with temperature – turning points of oscillator are zero points of velocity:

$$\Theta + \Psi \propto \cos(ks); \quad v_\gamma \propto \sin(ks)$$

- Polarization peaks are at troughs of temperature power

Cross Correlation

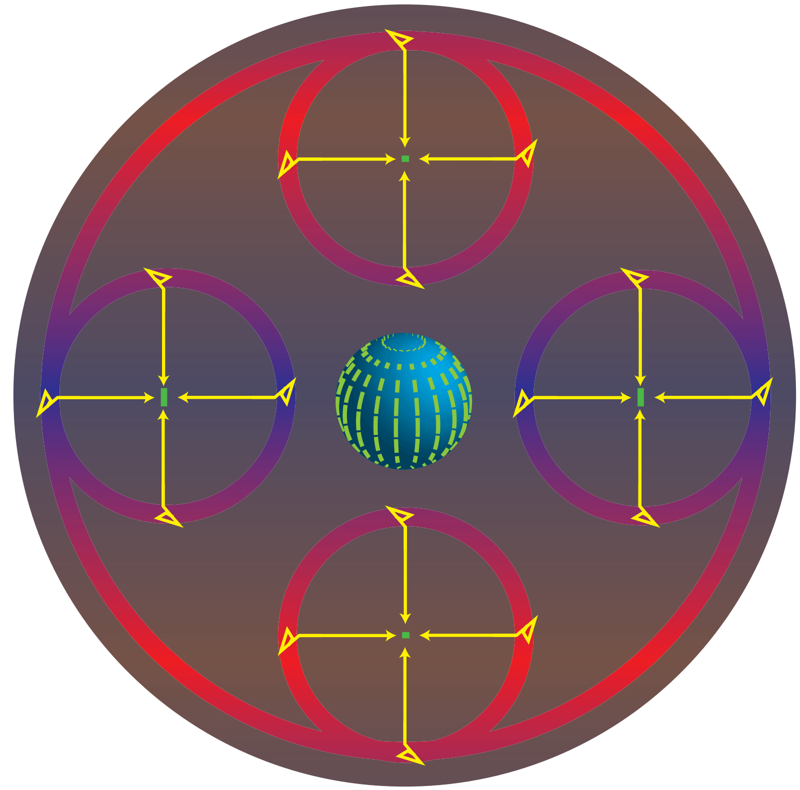
- Cross correlation of temperature and polarization

$$(\Theta + \Psi)(v_\gamma) \propto \cos(ks)\sin(ks) \propto \sin(2ks)$$

- Oscillation at twice the frequency
- Correlation: radial or tangential around hot spots
- Partial correlation: easier to measure if polarization data is noisy, harder to measure if polarization data is high S/N or if bands do not resolve oscillations
- Good check for systematics and foregrounds
- Comparison of temperature and polarization is proof against features in initial conditions mimicking acoustic features

Reionization

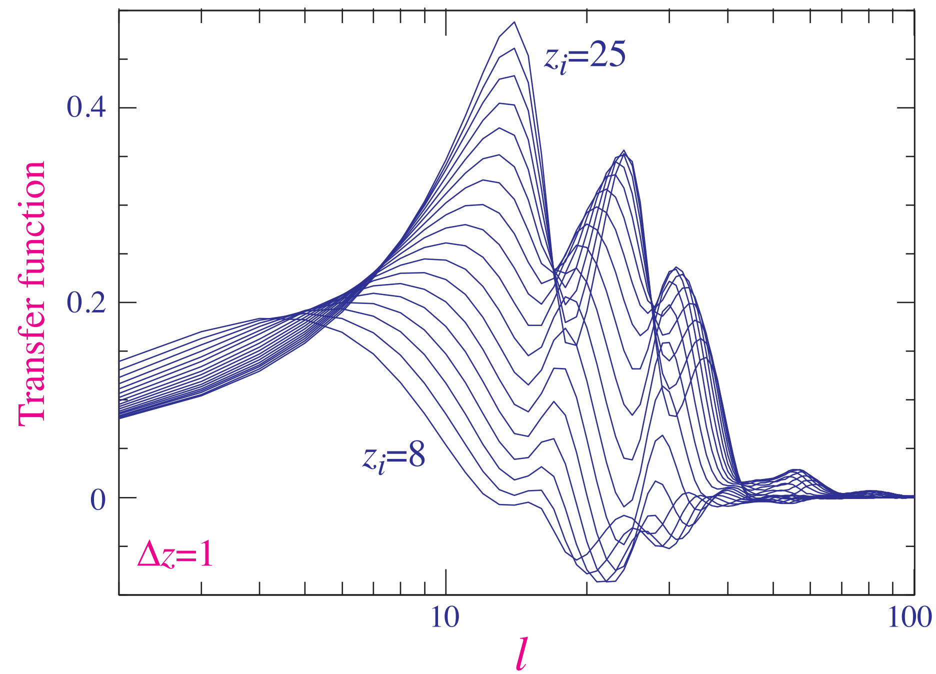
- Reionization causes rescattering of radiation
- Suppresses temperature anisotopy as $e^{-\tau}$ and changes interpretation of amplitude to $A_s e^{-2\tau}$
- Electron sees temperature anisotropy on its recombination surface



- For wavelengths that are comparable to the horizon at reionization, a quadrupole moment
- Rescatters to a linear polarization that is correlated with the Sachs-Wolfe temperature anisotropy

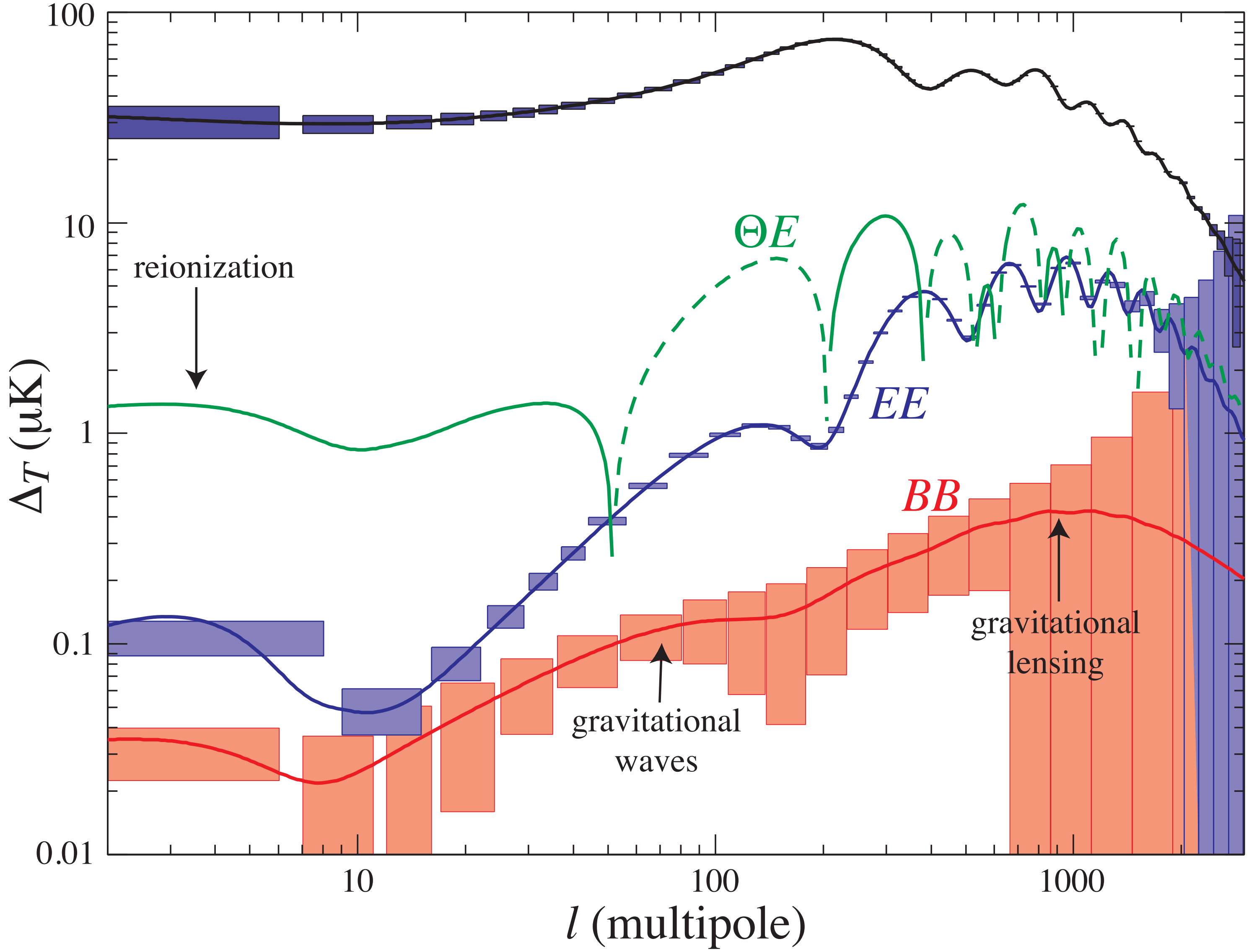
Reionization

- Amplitude of C_ℓ^{EE} depends mainly on τ
- Shape of C_ℓ^{EE} depends on reionization history
- Horizon at earlier epochs subtends a smaller angle, higher multipole peak



- Precision measurements can constrain the reionization history to be either low or high z dominated

Polarized Landscape



Gravitational Waves

- Gravitational wave amplitude satisfies Klein-Gordon equation $(K = 0)$, same as scalar field

$$\frac{d^2 h_{+,\times}}{dt^2} + 3H\frac{dh_{+,\times}}{dt} + \frac{k^2}{a^2}h_{+,\times} = 0\,.$$

- Acquires quantum fluctuations in same manner as ϕ. Lagrangian sets the normalization

- Scale-invariant gravitational wave amplitude

$$\Delta^2_{+,\times} = 16\pi G\frac{H^2}{(2\pi)^2}$$

- Gravitational wave power $\propto H^2 \propto V \propto E_i^4$ where E_i is the energy scale of inflation

Gravitational Waves

- Tensor-scalar ratio is therefore generally small

$$r \equiv 4\frac{\Delta_+^2}{\Delta_{\mathcal{R}}^2} = 16\epsilon_H$$

- Tensor tilt:

$$\frac{d\ln\Delta_+^2}{d\ln k} \equiv n_T = 2\frac{d\ln H}{d\ln k} = -2\epsilon_H$$

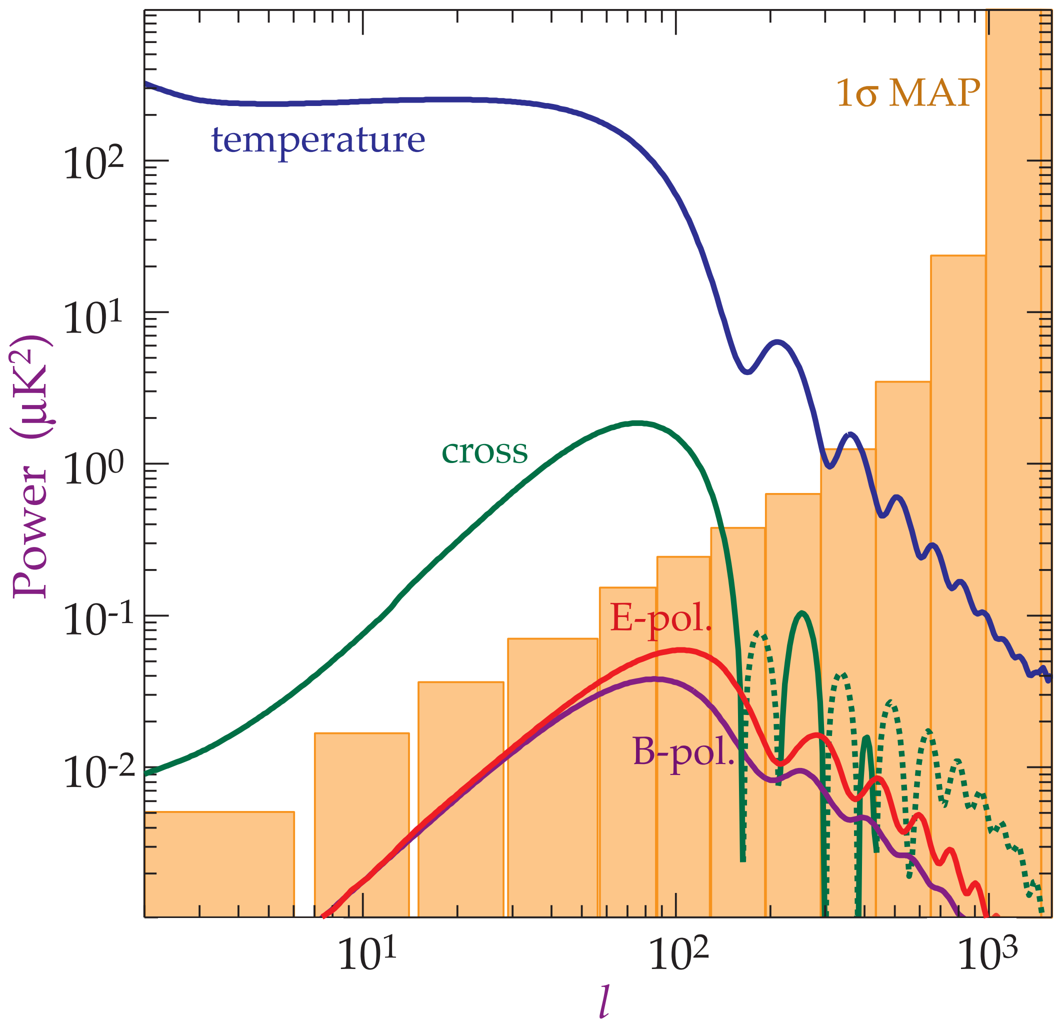
- Consistency relation between tensor-scalar ratio and tensor tilt

$$r = 16\epsilon = -8n_T$$

- Measurement of scalar tilt and gravitational wave amplitude constrains inflationary model in the slow roll context
- Comparision of tensor-scalar ratio and tensor tilt tests the idea of slow roll itself

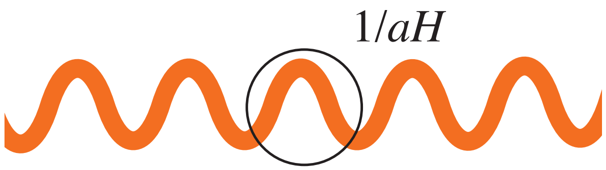
Observability

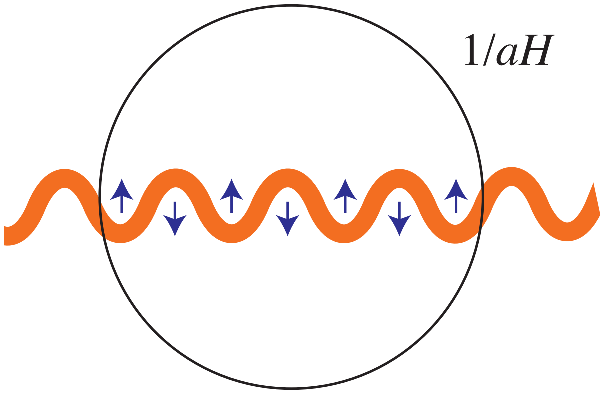
- Gravitational waves from inflation can be measured via its imprint on the polarization of the CMB



Tensor Power

- Gravitational waves obey
  a Klein-Gordon like equation
- Like inflation, perturbations generated
  by quantum fluctuations during inflation
- Freeze out at horizon crossing
  during inflation an amplitude
  that reflects the energy scale of inflation

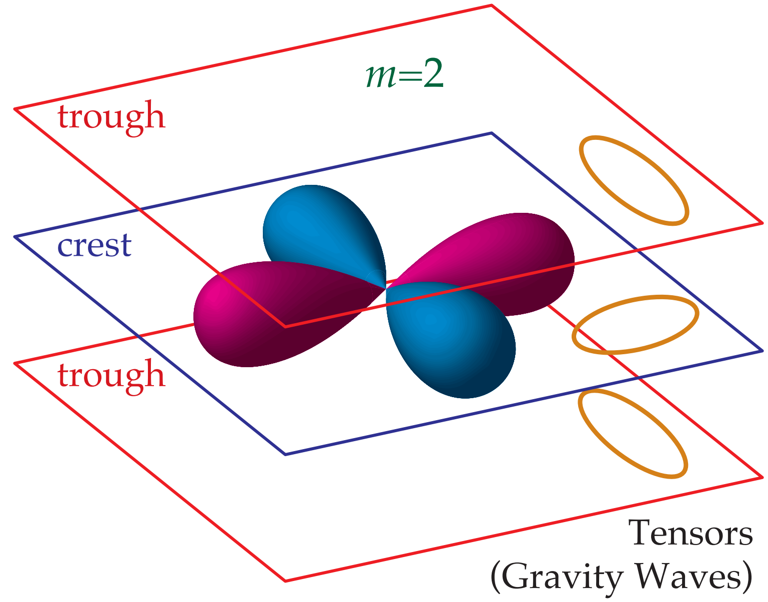




$$\Delta^2_{+,\times} = \frac{H^2}{2\pi^2 M_{\rm Pl}^2} \propto E_i^4$$

- Gravitational waves remain frozen outside the horizon at constant amplitude
- Oscillate inside the horizon and decay or redshift as radiation

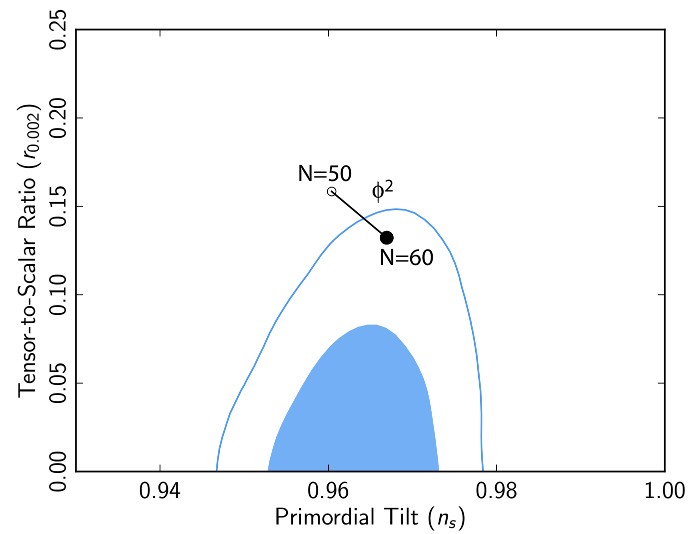
Tensor Quadrupoles



- Changing transverse-traceless distortion of space creates a quadrupole CMB anisotropy much like the distortion of test ring of particles
- As the tensor mode enters the horizon it imprints a quadrupole temperature $\ell = 2, m = \pm 2$ in plane wave coordinates $\mathbf{k} \parallel \mathbf{z}$
- Modes that cross before recombination: effect erased by rescattering $e^{-\tau}$ due to its isotropizing effect
- Modes that cross after recombination: integrate contributions along the line of sight - tensor ISW effect

Tensor Temperature Power Spectrum

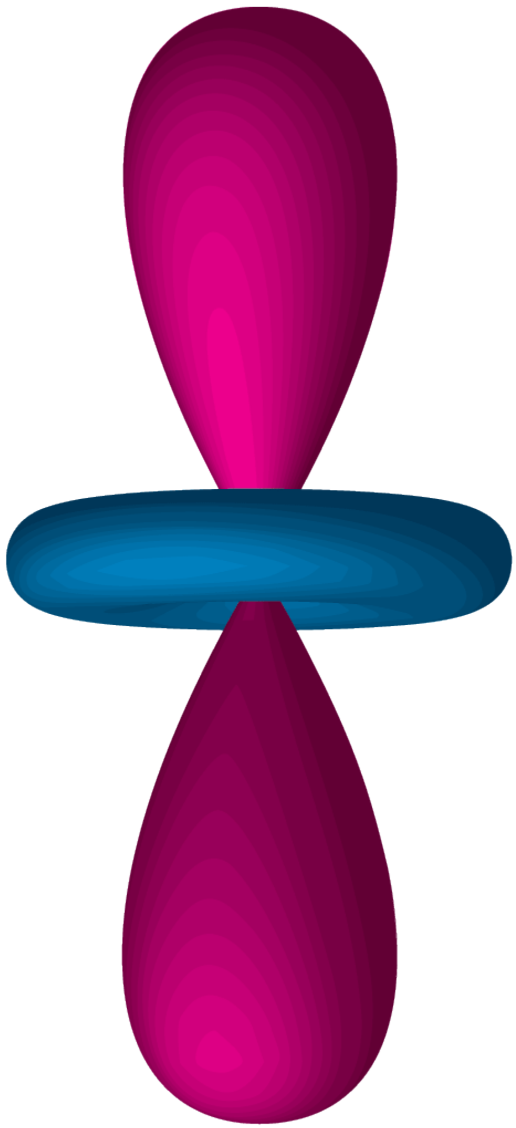
- Resulting spectum, near scale invariant out to horizon at recombination $\ell < 100$
- Suppressed on smaller scales or higher multipoles $\ell > 100$, weakly degenerate with tilt

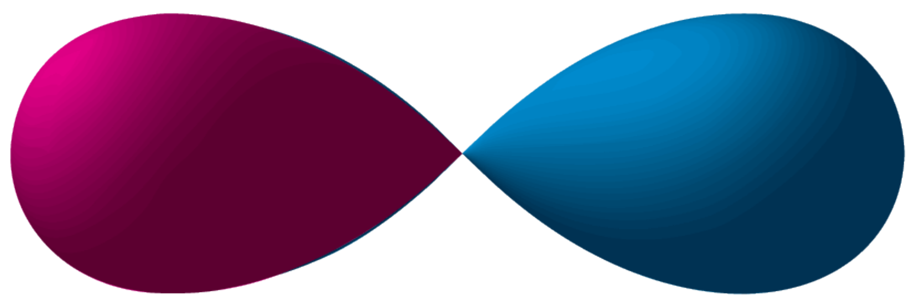


- When added to scalar spectrum, enhances large scale anisotropy over small scale
- Shape of total temperature spectrum can place tight limit $r < 0.1$
- Non-detection of B modes by BICEP limits $r < 0.03$
- Rules out monomial potentials $V \propto \phi^p$, including $p = 2$
- Allowed models have some scale so that potential is flatter on CMB scales

Polarization

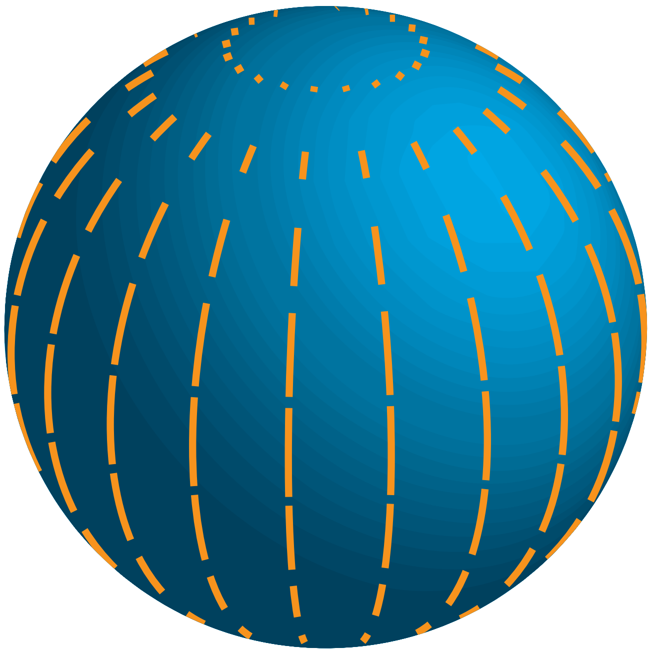
- Scalar vs tensor quadrupole viewed on sky



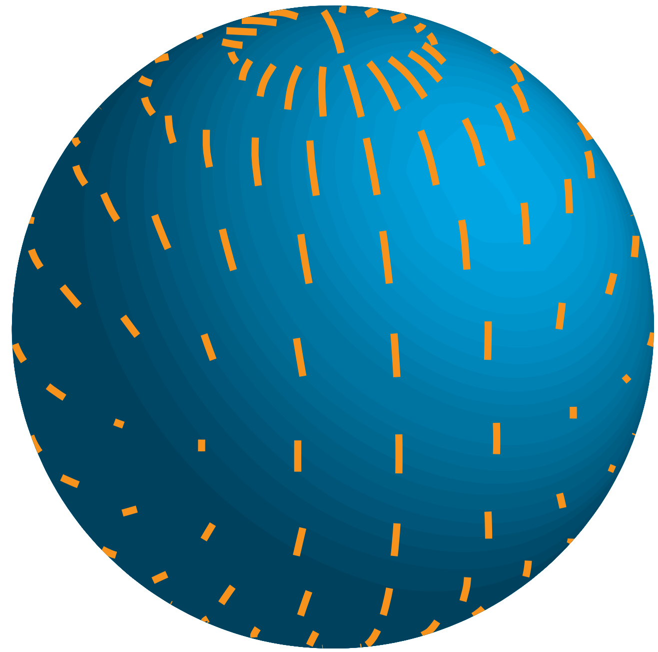


Polarization

- Scalars vs. tensors: E vs B



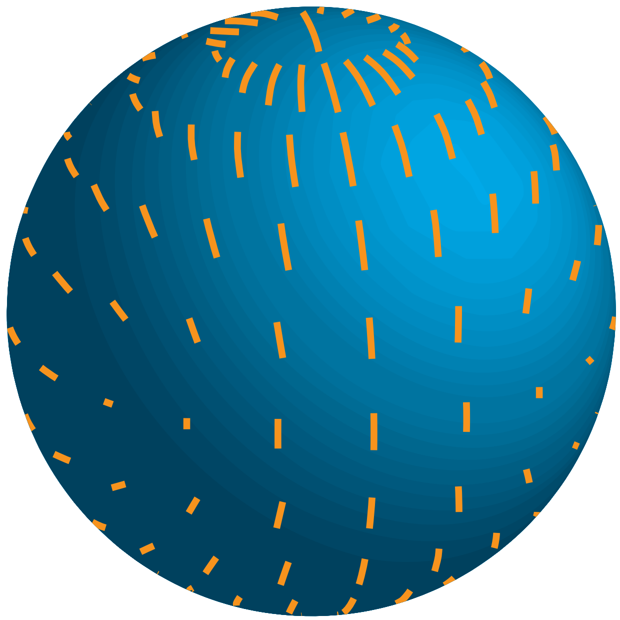
Scalars m=0



Tensors m=2

Tensor Polarization Power Spectrum

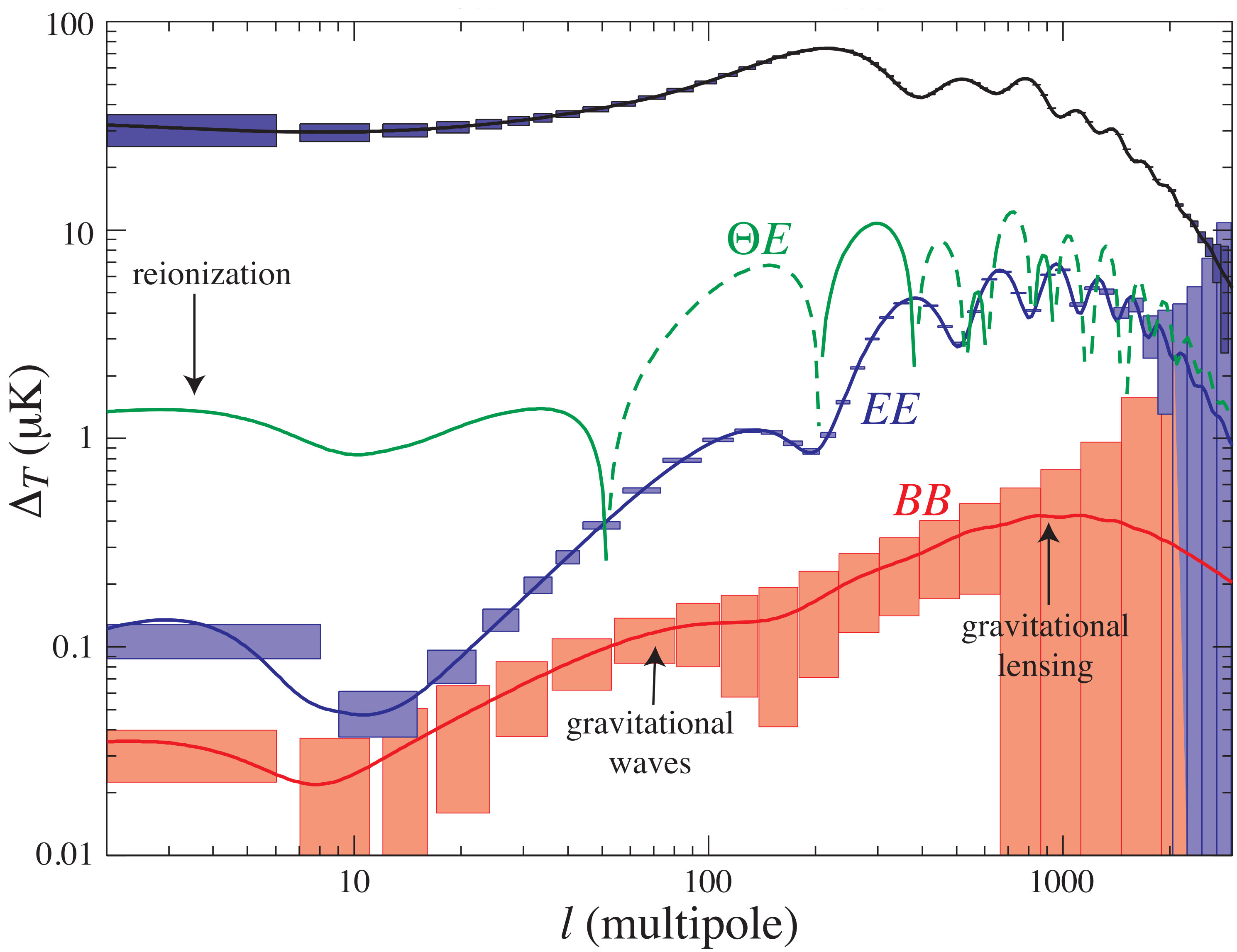
- Polarization of gravitational wave determines the quadrupole temperature anisotropy
- Scattering of quadrupole temperature anisotropy generates linear polarization aligned with cold lobe



- Direction of CMB polarization is therefore determined by gravitational wave polarization rather than direction of wavevector
- B-mode polarization when the amplitude is modulated by the plane wave
- Requires scattering: two peaks - horizon at recombination and reionization

Polarized Landscape

- Low ℓ gravitational wave B modes under the lensing B modes



Tensor Polarization Power Spectrum

- Measuring B-modes from gravitational waves determines the energy scale of inflation

$$\Delta B_{\rm peak} \approx 0.024 \left(\frac{E_i}{10^{16} \rm GeV} \right)^2 \mu \rm K$$

- Also generates E-mode polarization which, like T, is a consistency check but like T falls below detectability below $r \sim 0.1$

Large Scale Structure and Dark Energy

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Matter Evolution

- Relative to inflationary initial curvature power spectrum

$$\frac{k^3}{2\pi^2} P_{\mathcal{R}}(k) = A_S \left(\frac{k}{k_{\rm norm}} \right)^{n_S - 1}$$

transfer function $T(k)$ defines the subhorizon evolution which is influenced by pressure effects during radiation domination

- Normalize to the matter dominated expectation and take $\Phi = [3G(a)/5]\,\mathcal{R}$ where $G(a)$ is the modification to the growth rate of Φ due to the dark energy and curvature

$$\Phi(a, k) = \frac{3}{5} G(a) T(k) \mathcal{R}(0, k)$$

Matter Evolution

- Grouping the clustering matter as Ω_m, the Poisson equation converts curvature power spectrum and density fluctuation $k^2\Phi = 4\pi G a^2 \rho_m \Delta$ (where Δ is the density perturbation on comoving slicing)

$$\frac{k^3}{2\pi^2} P_\Delta(k) = \frac{4}{25} A_S \left(\frac{G(a)a}{\Omega_m} \right)^2 \left(\frac{k}{H_0} \right)^4 \left(\frac{k}{k_{\rm norm}} \right)^{n_S - 1} T^2(k)$$

Transfer Function

- Transfer function transfers the initial Newtonian curvature to its value today (linear response theory)

$$T(k) = \frac{\Phi(k, a=1)}{\Phi(k, a_{\rm init})} \frac{\Phi(k_{\rm norm}, a_{\rm init})}{\Phi(k_{\rm norm}, a=1)}$$

- Conservation of curvature : Newtonian curvature is a constant when stress perturbations are negligible: above the horizon during radiation and dark energy domination, on all scales during matter domination
- When stress fluctuations dominate, perturbations are stabilized by the Jeans mechanism
- Poisson equation: Newtonian curvature, comoving density perturbation $\Delta \equiv (\delta\rho/\rho)_{\rm com}$ implies Φ decays

$$(k^2 - 3K)\Phi = 4\pi G\rho\Delta \sim \eta^{-2}\Delta$$

Transfer Function

- Freezing of Δ at its horizon crossing value $\Delta_H \sim \Phi_{\rm init}$ stops at $\eta_{\rm eq}$

$$\Phi \sim (k\eta_{\rm eq})^{-2}\Delta_H \sim (k\eta_{\rm eq})^{-2}\Phi_{\rm init}$$

- Transfer function has a k^{-2} fall-off beyond $k_{\rm eq} \sim \eta_{\rm eq}^{-1}$
- Small correction since growth with a smooth radiation component is logarithmic not frozen
- Transfer function is a direct output of an Einstein-Boltzmann code

Fitting Function

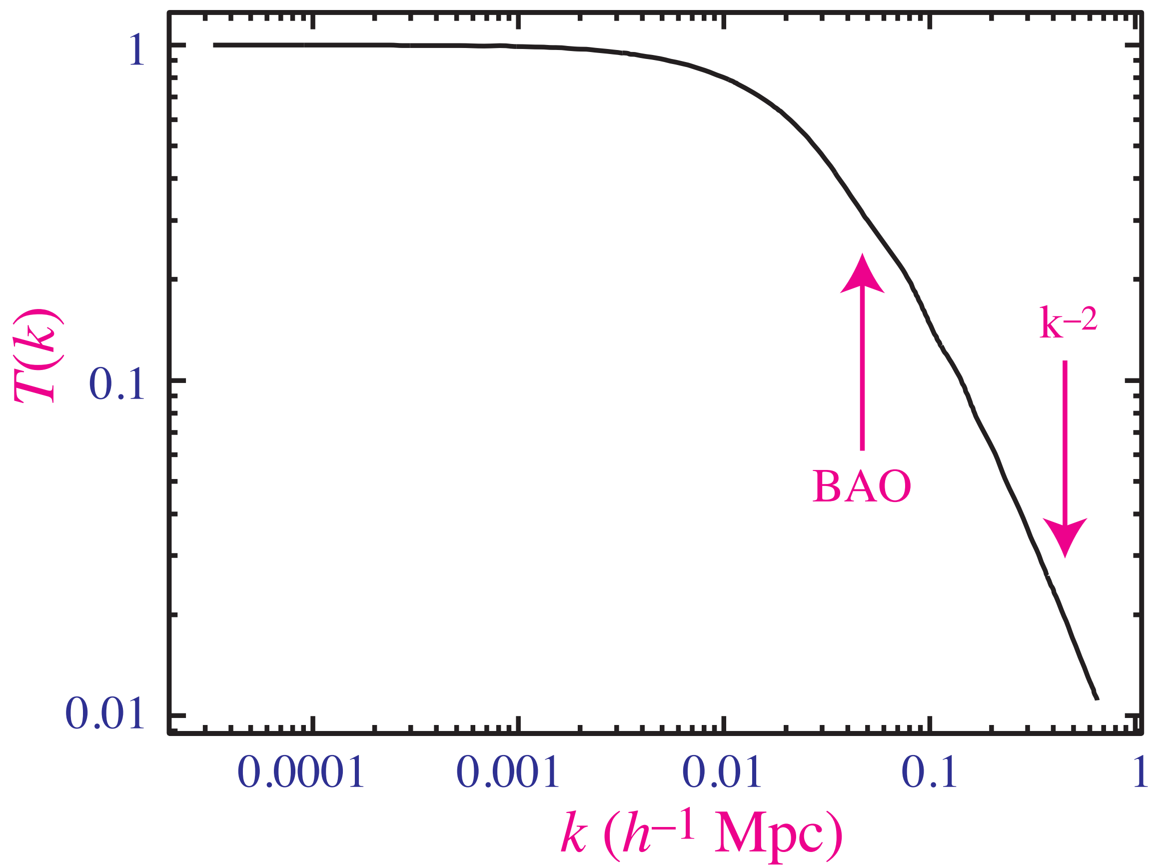
- Alternately accurate fitting formula exist, e.g. pure CDM form:

$$
\begin{aligned}
T(k(q)) &= \frac{L(q)}{L(q) + C(q)q^2} \\
L(q) &= \ln(e + 1.84q) \\
C(q) &= 14.4 + \frac{325}{1 + 60.5q^{1.11}} \\
q &= k/\Omega_m h^2 \mathrm{Mpc}^{-1} (T_{\rm CMB}/2.7K)^2
\end{aligned}
$$

- In h Mpc^{-1}, the critical scale depends on $\Gamma \equiv \Omega_m h$ also known as the shape parameter

Transfer Function

- Numerical calculation



Dark Matter and the Transfer Function

- Baryons caught up in the acoustic oscillations of the CMB and impart acoustic wiggles to the transfer function. Density enhancements are produced kinematically through the continuity equation $\delta_b \sim (k\eta)v_b$ and hence are out of phase with CMB temperature peaks
- Dissipation of the acoustic oscillations eliminates both the CMB and baryon perturbations – known as Silk damping for the baryons. This suppression and the general fact that baryons are caught up with photons was one of the main arguments for CDM
- Neutrino dark matter suffers similar effects and hence cannot be the main component of dark matter in the universe
- More generally beyond cold dark matter (warm, self-interacting, fuzzy) small scale structure can differ appreciably and still be consistent with current data

Massive Neutrinos

- Relativistic stresses of a light neutrino slow the growth of structure
- From a radiative transfer standpoint, this is due to the free streaming of neutrinos, like the free streaming of photons after recombination
- Neutrino species with cosmological abundance contribute to matter as $\Omega_\nu h^2 = \sum m_\nu / 94\text{eV}$, suppressing power as $\Delta P/P \approx -8\Omega_\nu/\Omega_m$
- Current data from CMB lensing has caused some controversy as to whether "negative neutrino mass" is required to invert this suppression

Growth Function

- Same physics applies to the dark energy dominated universe
- Under the dark energy sound horizon or Jeans scale, dark energy density frozen. Potential decays at the same rate for all scales

$$G(a) = \frac{\Phi(k_{\rm norm}, a)}{\Phi(k_{\rm norm}, a_{\rm init})} \qquad ' \equiv \frac{d}{d\ln a}$$

- Continuity + Euler + Poisson

$$G'' + \left(1 - \frac{\rho''}{\rho'} + \frac{1}{2}\frac{\rho_c'}{\rho_c}\right) G' + \left(\frac{1}{2}\frac{\rho_c' + \rho'}{\rho_c} - \frac{\rho''}{\rho'}\right) G = 0$$

where ρ is the Jeans unstable matter and $\rho_c = 3H^2/8\pi G$

Dark Energy Growth Suppression

- Consider the unstable matter to be $\rho = \rho_m \propto a^{-3}$ and smooth dark energy:

$$\frac{d^2 G}{d \ln a^2} + \left[\frac{5}{2} - \frac{3}{2} w(z)\Omega_{\rm de}(z)\right] \frac{dG}{d\ln a} + \frac{3}{2}[1 - w(z)]\Omega_{\rm de}(z) G = 0\,,$$

  where $w \equiv p_{\rm de}/\rho_{\rm de}$ and $\Omega_{\rm de} \equiv \rho_{\rm de}/(\rho_m + \rho_{\rm de})$ with initial conditions $G = 1$, $dG/d\ln a = 0$

- Quintessence, a scalar field with sound speed $c_s^2 = 1$ in a potential is a candidate for smooth dark energy

$$\Box\phi = -dV/d\phi \quad \rightarrow \quad \ddot{\phi} + 2aH\dot{\phi} + k^2\phi = -a^2 dV/d\phi$$

- Initially frozen by Hubble drag with $w_{\rm de} = -1$ and starts rolling as slope overcomes the decreasing Hubble friction $w_{\rm de} > -1$
- Cannot have $w_{\rm de} < -1$ or "phantom dark energy"

Current Amplitude, σ_8, S_8

- For ΛCDM ($w_{\rm de} = -1$) with neutrinos below free streaming scale

$$G_0 = \left(\frac{5}{2} \Omega_m \int_0^1 \frac{{\rm d}a}{(aH(a)/H_0)^3} \right) \times \left(1 - 0.014 \frac{\sum m_\nu}{0.06{\rm eV}} \right) .$$

or approximately

$$G_0 \approx 0.76 \left(\frac{\Omega_m}{0.27} \right)^{0.236} \left(1 - 0.014 \frac{\sum m_\nu}{0.06{\rm eV}} \right) .$$

- Using this growth factor:

$$\sigma_8 \approx \left(\frac{A_s}{3.135 \times 10^{-9}} \right)^{1/2} \left(\frac{\Omega_b h^2}{0.024} \right)^{-0.272} \left(\frac{\Omega_m h^2}{0.14} \right)^{0.513} \times (3.123h)^{(n_s-1)/2} \left(\frac{h}{0.72} \right)^{0.698} \left(\frac{G_0}{0.76} \right) ,$$

and $S_8 \equiv \sigma_8 (\Omega_m/0.3)^{1/2}$

Velocity field

- Continuity gives the velocity from the density field as

$$v = -\dot{\Delta}/k = -\frac{aH}{k}\frac{d\Delta}{d\ln a}$$
$$= -\frac{aH}{k}\Delta\frac{d\ln(aG)}{d\ln a}$$

- In a ΛCDM model or open model $d\ln(aG)/d\ln a \approx \Omega_m^{0.6}$
- Measuring both the density field and the velocity field (through distance determination and redshift) allows a measurement of Ω_m
- Practically one measures $\beta = \Omega_m^{0.6}/b$ where b is a bias factor for the tracer of the density field, i.e. with galaxy numbers $\delta n/n = b\Delta$
- Can also measure this factor from the redshift space power spectrum - the Kaiser effect where clustering in the radial direction is apparently enhanced by gravitational infall

Gravitational Lensing

- In general relativity, masses curve space and bend the trajectory of photons - for this discussion lets restore the different units of t and x by restoring c - but note that is now does not represent the coordinate speed of light
- Newtonian approximation to the line element, neglecting the expansion (or in conformal coordinates)

$$ds^2 = -(1 + 2\Psi/c^2)c^2 dt^2 + (1 + 2\Phi/c^2)dx^2$$

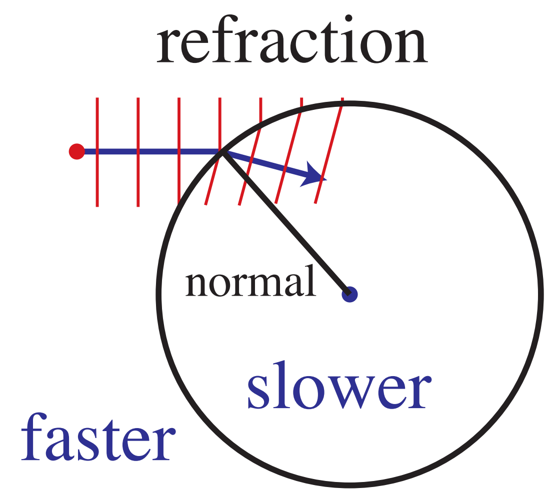
- Photons travel on null geodesics ($ds^2 = 0$) - so the coordinate speed of light is

$$v = \frac{dx}{dt} \approx c\frac{1 + \Psi/c^2}{1 + \Phi/c^2} \approx c(1 - 2\Phi/c^2)$$

Gravitational Lensing

- Coordinate speed of light slows
  in the presence of mass due to
  the warping of spacetime
  as quantified by the gravitational potential

  Can be modelled as an optics problem,
  defines an effective index of refraction

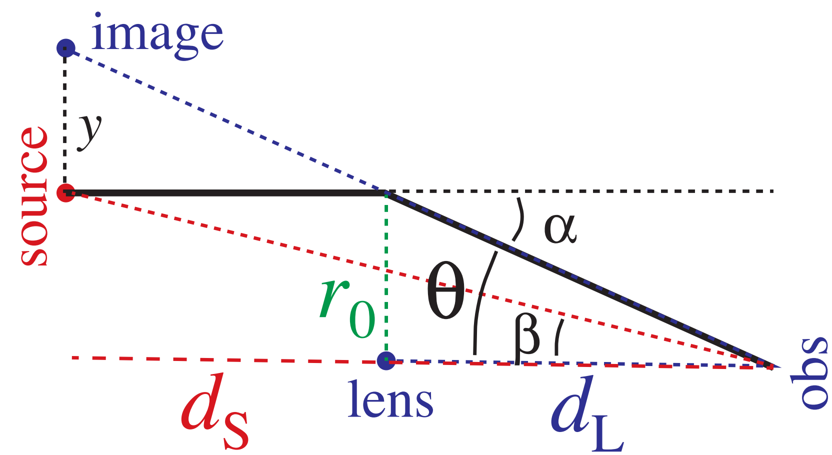


$$n = \frac{c}{v} = \left(1 - \frac{2\Phi}{c^2}\right)^{-1} \approx 1 + \frac{2\Phi}{c^2}$$

- As light passes by the object, the change in the index of refraction
  or delay of the propagation of wavefronts bends the trajectory

$$\nabla n = \frac{2}{c}\nabla\Phi = -\frac{2GM}{r^2c^2}\hat{\mathbf{r}}$$

Strong Gravitational Lensing



- Calculation would take the same form if we took a nonrelativistic particle of mass m and used Newtonian mechanics - general relativity just doubles it the deflection for light due to space curvature
- Deflection is small so integrate the transverse ($\perp$) deflection on the unperturbed trajectory

$$\alpha = -\int_{-\infty}^{\infty} dx \nabla_\perp n = \int_{-\infty}^{\infty} dx \frac{2GMr_0}{\left(r_0^2 + x^2\right)^{3/2} c^2} = \frac{4GM}{r_0 c^2}$$

Lens Equation

- Given the thin lens deflection formula, the lens equation follows from simple geometry
- Solve for the image position θ with respect to line of sight. Small angle approximation

$$y \approx (d_S - d_L)\alpha \approx d_S(\theta - \beta)$$

- Substitute in deflection angle

$$(d_S - d_L)\frac{4GM}{r_0 c^2} \approx d_S(\theta - \beta)$$

- Eliminate $r_0 = d_L \sin\theta \approx d_L\theta$
- Solve quadratic equation in θ for the multiply lensed images

From Point Lens to Continuous

- Deflection angle for a point lens across small distance

$$\theta - \beta = 2 \int dx \frac{d_S - d_L}{d_S} \nabla_\perp \Phi$$

- Promote to comoving coordinates and allow the variation of Φ to occur across cosmological scales

$$\theta - \beta = 2 \int d\eta \frac{D_S - D_L}{D_S} \nabla_\perp \Phi(D_L \hat{\mathbf{n}}, \eta)$$

  note that converting for point mass $\Phi = GM/r = GM(1+z)/R$ is equivalent to the "redshifted mass" $\mathcal{M} = M(1+z)$

- Convert the transverse spatial derivative at the distance D_L to an angular derivative

$$\nabla_{\hat{n}} = D_L \nabla_\perp$$

  and define the deflection potential

$$\theta - \beta = \nabla_{\hat{\mathbf{n}}} \phi(\hat{\mathbf{n}})$$

Gravitational Lensing

- Gravitational potentials along the line of sight $\hat{\mathbf{n}}$ to some source at comoving distance D_S lens the images according to (flat universe)

$$\phi(\hat{\mathbf{n}}) = 2\int dD_L \frac{D_S - D_L}{D_L D_S} \Phi(D_L\hat{\mathbf{n}}, \eta(D_L))$$

  remapping image positions as

$$\hat{\mathbf{n}}^I = \hat{\mathbf{n}}^S + \nabla_{\hat{\mathbf{n}}}\phi(\hat{\mathbf{n}})$$

- Since absolute source position is unknown, use image distortion defined by the Jacobian matrix

$$\frac{\partial n_i^I}{\partial n_j^S} = \delta_{ij} + \psi_{ij}$$

Weak Lensing

- Small image distortions described by the convergence κ and shear components (γ_1, γ_2)

$$\psi_{ij} = \begin{pmatrix} \kappa - \gamma_1 & -\gamma_2 \\ -\gamma_2 & \kappa + \gamma_1 \end{pmatrix}$$

where $\nabla_{\hat{\mathbf{n}}} = D\nabla$ and

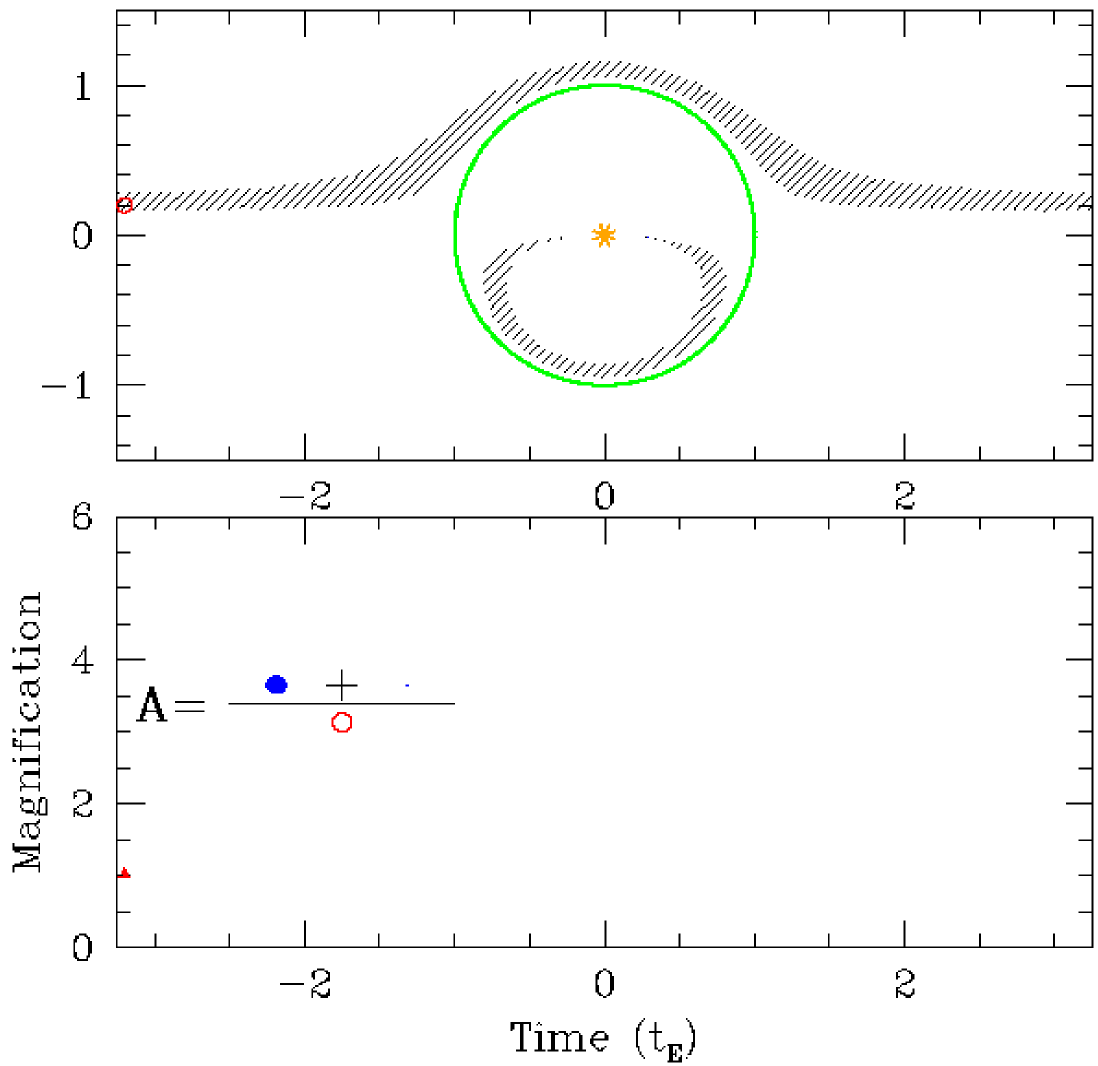
$$\psi_{ij} = 2 \int dD_L \frac{D_L(D_S - D_L)}{D_S} \nabla_i \nabla_j \Phi(D_L \hat{\mathbf{n}}, \eta(D_L))$$

- In particular, through the Poisson equation the convergence (measured from shear) is simply the projected mass

$$\kappa = \frac{3}{2} \Omega_m H_0^2 \int dD_L \frac{D_L(D_S - D_L)}{D_S} \frac{\Delta(D_L \hat{\mathbf{n}}, \eta(D_L))}{a}$$

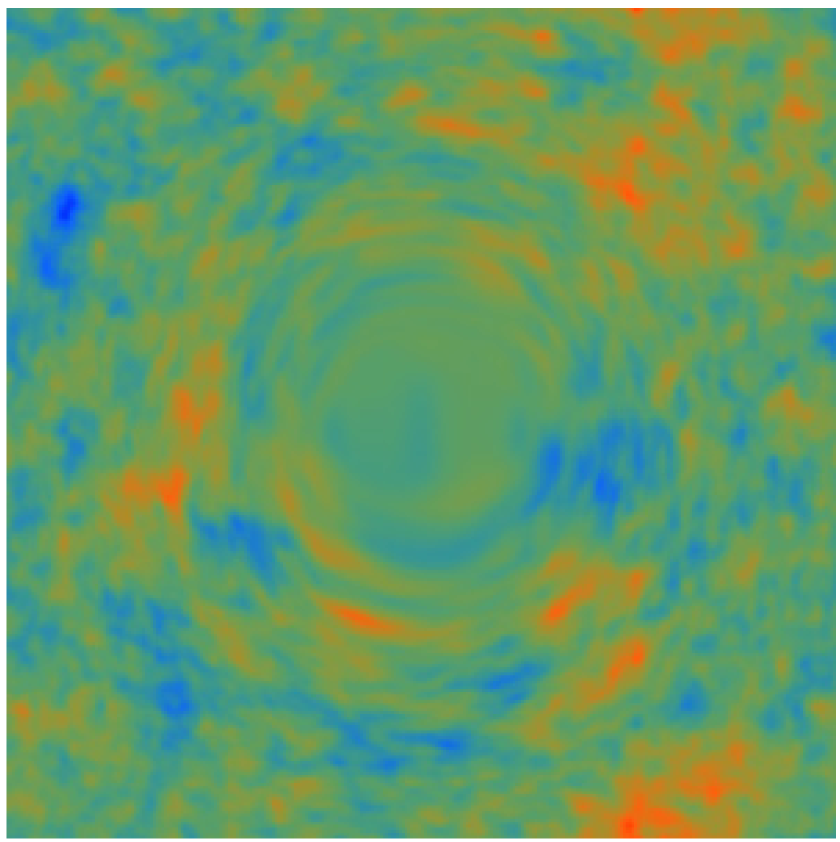
Point lens

- Two images on opposite side of lens (magnified, sheared and time delayed)



CMB Lensing

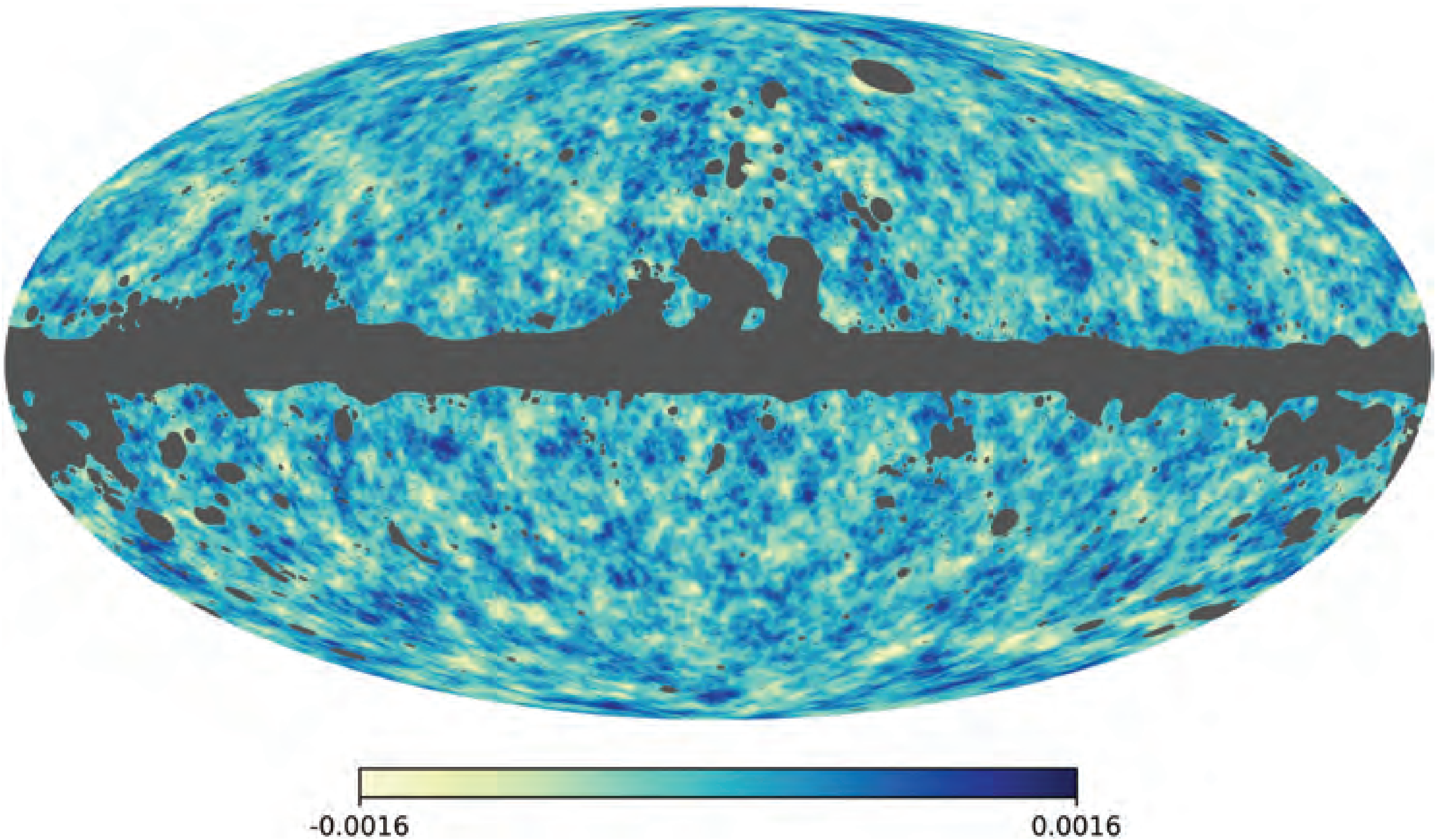
- Temperature fluctuations experience magnification and shear allowing mass reconstruction
- CMB lensing by an unrealistically large lens



CMB Lensing

- Gravitational Lensing measures projected mass
- Planck CMB lensing map

lensing



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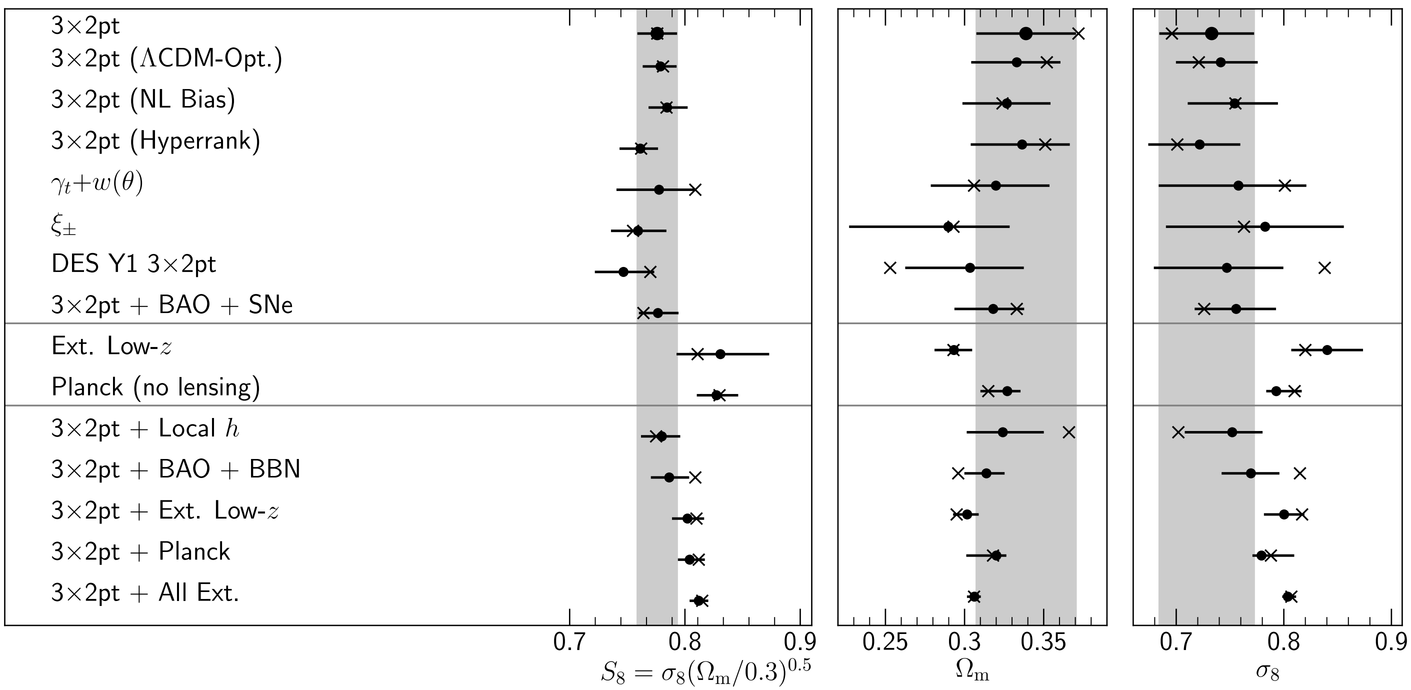
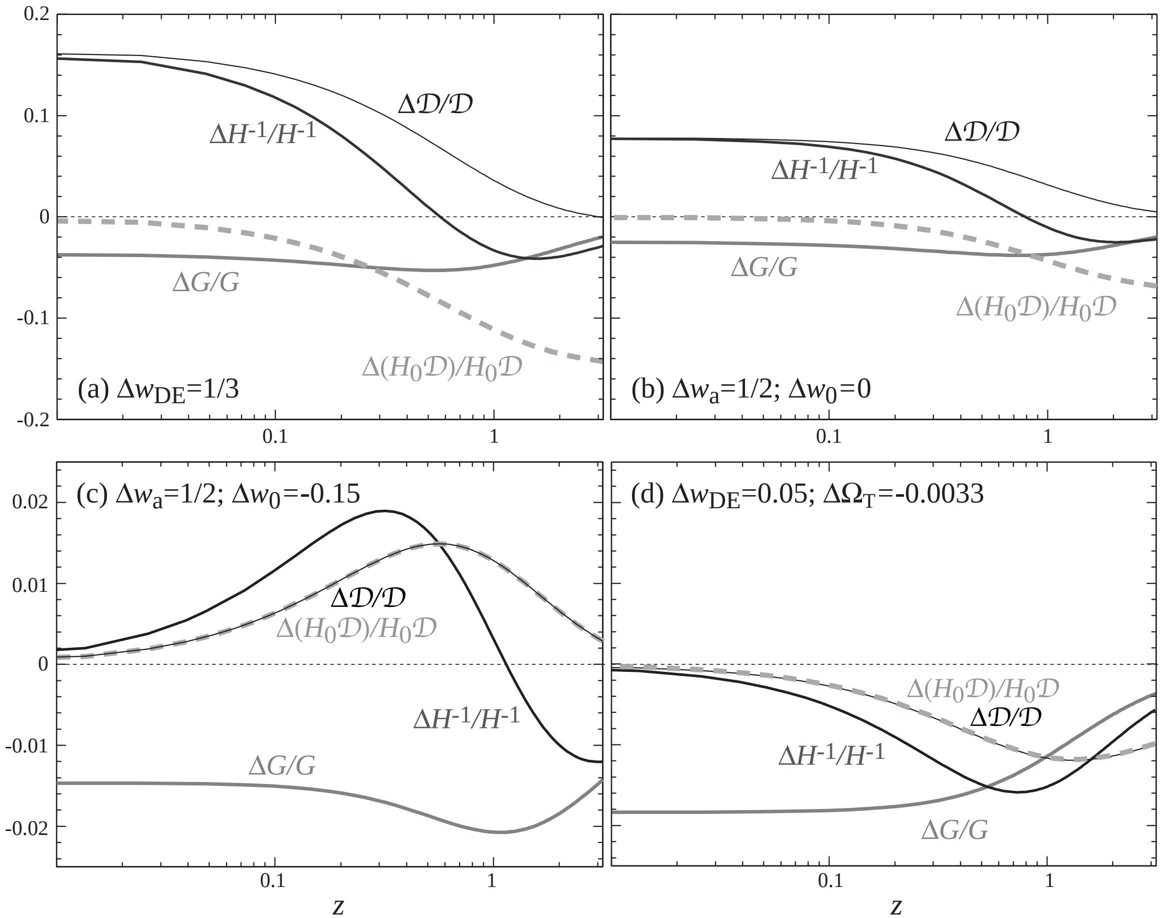
- Galaxy-Galaxy, Galaxy-Shear, Shear-Shear power spectra
- Infer galaxy bias and growth as well as halo occupation distribution
- Controversy as to whether S_8 is consistent with ΛCDM predictions from CMB and BAO



Dark Energy Observables

- Fixed high z, CMB $(\mathcal{D} = D_A)$



- Solutions to $D(z)$ tensions in SN, BAO predict matching deviations in structure

Tensions and their Resolutions

Words of wisdom:

- Cosmological tensions come and go
- Use them to learn what is compelling about the standard picture
- Use them to see how to break the standard picture – the assumptions behind a purported no-go
- Only become a true believer if you predict some new phenomena that is then observationally verified!