Astro 408

Set 1: Relativistic Perturbation Theory

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Evolution of Spacetime aka Universe

- General Relativity

  Dynamical spacetime: no pre-fixed spacetime coordinates

  Covariant: no preferred coordinate choice

- Conceptually: how do we think about the evolution of the Universe when there is no prefixed or preferred choice of time?
- Practically: how do we exploit the coordinate freedom of GR to simplify that task

Covariant Perturbation Theory

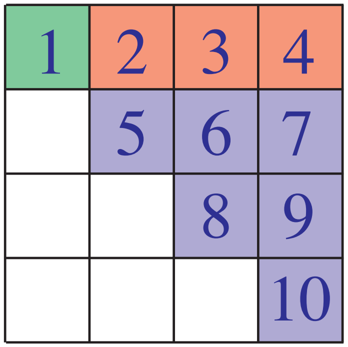
- Covariant = takes same form in all coordinate systems
- Invariant = takes the same value in all coordinate systems
- Fundamental equations are covariant: Einstein equations, covariant conservation of stress-energy tensor:

$$\begin{aligned} G_{\mu\nu} &= 8\pi G T_{\mu\nu} \\ \nabla_\mu T^{\mu\nu} &= 0 \end{aligned}$$

- Components such as velocity, curvature etc are *not invariant* under a coordinate change - furthermore the same coordinates in the background can refer to different spacetime points
- Between any two fully specified coordinates, Jacobian $\partial x^\mu / \partial \tilde{x}^\nu$ is invertible - so perturbations in given gauge can be written in a covariant manner in terms of perturbations in an arbitrary gauge: called “gauge invariant” variables

Covariant Perturbation Theory

- In evolving perturbations we inevitably break explicit covariance by evolving conditions forward in a given time coordinate
- Retain implicit covariance by allowing the freedom to choose an arbitrary time slicing and spatial coordinates threading constant time slices
- Exploit covariance by choosing the specific slicing and threading (or "gauge") according to what best matches problem
- Preserve general covariance by keeping all free variables: 10 for each symmetric 4×4 tensor



1	2	3	4
	5	6	7
		8	9
			10

Covariant Objectives

- Characterize the 10 metric components
- Pick a preferred choice of temporal and spatial coordinates
- Retain general covariance: works for any choice of preferred coordinates
- Each set of preferred coordinates is represented fully covariantly and can be evaluated in any other set
- Start with the idea that the preferred choice is linked to the rest frame of an observer with 4-velocity V^μ
- Generalize to any timelike vector:

  A(rnowitt)D(eser)M(isner) 3+1 split
- ADM is a non-perturbative construction – after general ADM description we'll simplify for perturbation theory to linearize fluctuations

ADM 3+1 Split

- Einstein equations dynamically evolve the spacetime: to solve the initial value problem choose a coordinate "scaffolding" that builds spacetime and its dynamical spatial metric as it evolves forward
- Define most general line element: lapse N, shift N^i, 3-metric h_{ij}

$$ds^2 = -N^2 d\phi^2 + h_{ij}(dx^i + N^i d\phi)(dx^j + N^j d\phi)$$

or equivalently the metric

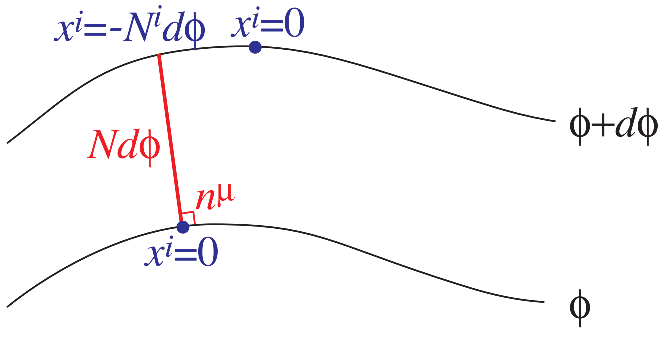
$$g_{00} = -N^2 + N^i N_i, \quad g_{0i} = h_{ij} N^j \equiv N_i, \quad g_{ij} = h_{ij}$$

and its inverse $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu$

$$g^{00} = -1/N^2, \quad g^{0i} = N^i/N^2, \quad g^{ij} = h^{ij} - N^i N^j/N^2$$

- Time coordinate $x^0 = \phi$ need not be proper time of an observer - suggestively labeled as scalar field → EFT of inflation, dark energy

ADM 3+1 Split



- Define the unit normal timelike vector $n_\mu n^\mu = -1$, orthogonal to constant time surfaces $n_\mu \propto \partial_\mu \phi$
- In preferred coordinates for time, it points in the time direction $n_\mu \propto (1,0,0,0)$

$$n_\mu = (-N, 0, 0, 0), \quad n^\mu = (1/N, -N^i/N)$$

where we have used $n^\mu = g^{\mu\nu} n_\nu$ to set the normalization

- Interpretation: lapse of proper time along normal, shift of spatial coordinates with respect to normal
- In GR (and most scalar-tensor EFT extensions), the lapse and shift are non-dynamical and just define the coordinates or gauge
- Dynamics in evolving the spatial metric forwards

ADM 3+1 Split

- Projecting 4D tensors onto the normal direction utilizes $n^\mu n_\nu$, e.g.

$$-n^\mu n_\nu V^\nu$$

- Projecting 4D tensors onto the 3D tensors involves the complement through the induced metric

$$h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu, \quad h^\mu{}_\nu V^\nu = (\delta^\mu{}_\nu + n^\mu n_\nu) V^\nu = V^\mu + n^\mu n_\nu V^\nu$$

  e.g. in the preferred slicing

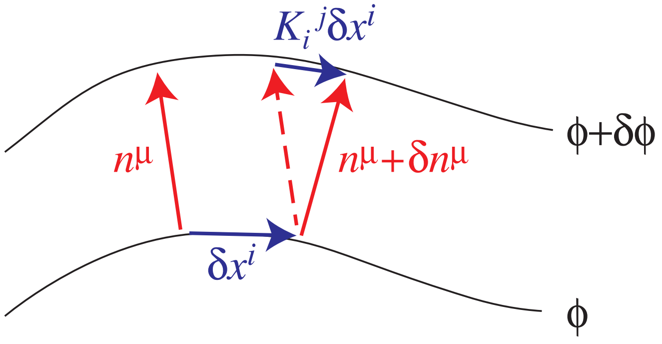
$$\tilde{V}^\mu = h^\mu{}_\nu V^\nu = (\delta^\mu{}_\nu + n^\mu n_\nu) V^\nu = (0, V^i + N^i V^0)$$

  whose spatial indices are raised an lowered by h_{ij}:
  $\tilde{V}_i = g_{i\nu} \tilde{V}^\nu = h_{ij} \tilde{V}^j$ since $h_{ij} = g_{ij}$ and $\tilde{V}^0 = 0$

- Notice also that if the preferred slicing is orthogonal to V^μ then $h^\mu{}_\nu V^\nu = 0$ and $V^i = -N^i V^0$ which defines comoving slicing

ADM 3+1 Split

- 3-surface embedded in 4D, so there is both an intrinsic curvature associated with h_{ij} and an extrinsic curvature which is the spatial projection of the gradient of n^μ



$$K_{\mu\nu} = h_\mu{}^\alpha h_\nu{}^\beta n_{\alpha;\beta}$$

- $K_{\mu\nu}$ symmetric since the antisymmetric projection (or vorticity) vanishes by construction since $n_\mu \propto \phi_{;\mu} = \phi_{,\mu}$

ADM 3+1 Split

- Likewise split the spacetime curvature ${}^{(4)}R$ into intrinsic and extrinsic pieces via Gauss-Codazzi relation

  $$ {}^{(4)}R = K_{\mu\nu}K^{\mu\nu} - (K_{\mu}{}^{\mu})^2 + {}^{(3)}R + 2(K_{\nu}{}^{\nu}n^{\mu} - n^{\alpha}n^{\mu}{}_{;\alpha})_{;\mu} $$

  Last term is total derivative so Einstein Hilbert action from Lagrangian $\mathcal{L} = {}^{(4)}R(M_{\rm pl}/2)$ same as keeping first three terms
- No explicit dependence on slicing and threading N, N^i - any preferred slicing is picked out by the matter distribution not by general relativity
- Beyond GR we can embed a preferred slicing by making the Lagrangian an explicit function of N - will return to this in the effective field theory of inflation, dark energy

ADM 3+1 Split

- Trace ${K_\mu}^\mu = {n^\mu}_{;\mu} \equiv \theta$ is expansion
- Avoid confusion with FRW notation for intrinsic curvature: ${}^{(3)}R = 6K/a^2$
- The anisotropic part is known as the shear

$$\sigma_{\mu\nu} = K_{\mu\nu} - \frac{\theta}{3} h_{\mu\nu}$$

- For the FRW background the shear vanishes and the expansion $\theta = 3H$

ADM 3+1 Split

- Fully decompose the 4-tensor $n_{\mu;\nu}$ by adding normal components

$$\begin{aligned} n_{\mu;\nu} &= K_{\mu\nu} - n_\mu n^\alpha h_\nu{}^\beta n_{\alpha;\beta} - h_\mu{}^\alpha n_\nu n^\beta n_{\alpha;\beta} + n_\mu n^\alpha n_\nu n^\beta n_{\alpha;\beta} \\ &= K_{\mu\nu} - h_\mu{}^\alpha n_\nu n^\beta n_{\alpha;\beta} = K_{\mu\nu} - n_\nu n^\beta n_{\mu;\beta} - n_\mu n^\alpha n_\nu n^\beta n_{\alpha;\beta} \\ &= K_{\mu\nu} - n_\nu n^\beta n_{\mu;\beta} = K_{\mu\nu} - a_\mu n_\nu \end{aligned}$$

  where we have used $[(n_\mu n^\mu)_{;\nu} = 0 \to n^\mu n_{\mu;\nu} = 0]$ and $h_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$

- Here the directional derivative of the normal along the normal or "acceleration" is

$$a_\mu = (n_{\mu;\beta}) n^\beta$$

- We shall see that in preferred coordinates, spatial component a_i involve the gradient of the lapse, i.e. GR analogue of Newtonian acceleration

ADM 3+1 Split

- Since

  $$K_{ij} = n_{i;j} = -\Gamma^{\mu}_{ij} n_{\mu} = \Gamma^{0}_{ij} N$$

  in terms of the ADM variables

  $$K_{ij} = \frac{1}{2N}(\partial_t h_{ij} - N_{j|i} - N_{i|j})$$

  where $|$ denotes the covariant derivative constructed from the 3-metric h_{ij}
- Extrinsic curvature acts like a "velocity" term for h_{ij} moving the metric from one slice to another with the coordinate freedom of the lapse and shift
- Initial value problem in GR: define h_{ij} and $\dot{h}_{ij}$ on the spacelike surface and integrate forwards, with lapse and shift defining the temporal and spatial coordinates

ADM 3+1 Split

- Beyond GR we can extend this logic by constructing a general theory with some scalar whose constant surfaces define the normal and the time coordinate - build the most general action that retains spatial diffeomorphism invariance out of the ADM geometric objects

  → EFT of inflation and dark energy: return to in applications/projects
- For linear perturbation theory in GR, ADM looks simpler since we can linearize metric fluctuations and take out the global scale factor in the spatial tensors for convenience $h_{ij} = a^2\gamma_{ij}$
- ADM language useful in defining the geometric meaning of gauge choices in defining the time slicing and spatial threading

Metric Perturbations

- ADM on the conformally rescaled metric $\tilde{g}_{\mu\nu}$ with $g_{\mu\nu} = a^2\tilde{g}_{\mu\nu}$, recall FRW background

$$\begin{aligned} d\tilde{s}^2 &= \tilde{g}_{\mu\nu}dx^\mu dx^\nu = -d\eta^2 + \gamma_{ij}dx^i dx^j \\ &= -d\eta^2 + dD^2 + D_A^2(d\theta^2 + \sin^2\theta d\phi^2) \end{aligned}$$

  where $D_A = R\sin(D/R)$ and $K = 1/R^2$

- Background lapse $\bar{N} = 1$, and shift $\bar{N}^i = 0$ so define perturbations $N = (1 + A)$, shift $N^i = -B^i$

$$\begin{aligned} \tilde{g}_{00} &= -(1 + 2A)\,, \\ \tilde{g}_{0i} &= -\gamma_{ij}B^j \equiv -B_i \end{aligned}$$

  where to linear order indices on 3-tensors raised and lowered by γ_{ij}

- This absorbs 1+3=4 free variables in the metric

Metric Perturbations

- Remaining 6 is in the spatial surfaces which we parameterize as

  $$\tilde{g}_{ij} = \gamma_{ij} + 2H_L\gamma_{ij} + 2H_{Tij}$$

  here (1) H_L a perturbation to the scale factor; (5) H_T^{ij} a trace-free distortion to spatial metric

- Curvature perturbation on the 3D slice, hereafter ∇^2 is the 3-Laplacian using covariant derivatives of 3-metric γ_{ij}

  $${}^{(3)}R = \frac{6K}{a^2} - \frac{4}{a^2}\left(\nabla^2 + 3K\right) H_L + \frac{2}{a^2}\nabla_i\nabla_j H_T^{ij}$$

  where recall that K characterizes the background intrinsic curvature

- Curvature perturbation is a 3-scalar in the ADM split and a Scalar in the SVT decomposition

Matter Tensor

- Likewise expand the matter stress energy tensor around a homogeneous density ρ and pressure p:

$$\begin{aligned}
T^0{}_0 &= -\rho - \delta\rho\,, \\
T^0{}_i &= (\rho + p)(v_i - B_i)\,, \\
T_0{}^i &= -(\rho + p)v^i\,, \\
T^i{}_j &= (p + \delta p)\delta^i{}_j + p\Pi^i{}_j\,,
\end{aligned}$$

- (1) $\delta\rho$ a density perturbation; (3) v_i a vector velocity, (1) δp a pressure perturbation; (5) Π_{ij} an anisotropic stress perturbation
- So far this is fully general and applies to any type of matter or coordinate choice including non-linearities in the matter, e.g. scalar fields, cosmological defects, exotic dark energy.

Counting Variables

20	Variables (10 metric; 10 matter)
-10	Einstein equations
-4	Conservation equations
$+4$	Bianchi identities
-4	Gauge (coordinate choice 1 time, 3 space)
6	Free Variables

- Without loss of generality these can be taken to be the 6 components of the matter stress tensor
- For the background, specify $p(a)$ or equivalently $w(a) \equiv p(a)/\rho(a)$ the equation of state parameter.

Homogeneous Einstein Equations

- Einstein (Friedmann) equations:

$$\left(\frac{1}{a}\frac{da}{dt}\right)^2 = -\frac{K}{a^2} + \frac{8\pi G}{3}\rho \quad [= \left(\frac{1}{a}\frac{\dot a}{a}\right)^2 = H^2]$$

$$\frac{1}{a}\frac{d^2 a}{dt^2} = -\frac{4\pi G}{3}(\rho + 3p) \quad [= \frac{1}{a^2}\frac{d}{d\eta}\frac{\dot a}{a} = \frac{1}{a^2}\frac{d}{d\eta}(aH)]$$

  so that $w \equiv p/\rho < -1/3$ for acceleration

- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot\rho}{\rho} = -3(1 + w)\frac{\dot a}{a}$$

  overdots are conformal time but equally true with coordinate time

Homogeneous Einstein Equations

- Counting exercise:

20	Variables (10 metric; 10 matter)
-17	Homogeneity and Isotropy
-2	Einstein equations
-1	Conservation equations
$+1$	Bianchi identities
1	Free Variables

without loss of generality choose ratio of homogeneous & isotropic component of the stress tensor to the density $w(a) = p(a)/\rho(a)$.

Acceleration Implies Negative Pressure

- Role of stresses in the background cosmology
- Homogeneous Einstein equations $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ imply the two Friedmann equations (flat universe, or associating curvature $\rho_K = -3K/8\pi G a^2$)

$$\begin{aligned}\left(\frac{1}{a}\frac{da}{dt}\right)^2 &= \frac{8\pi G}{3}\rho \\ \frac{1}{a}\frac{d^2a}{dt^2} &= -\frac{4\pi G}{3}(\rho + 3p)\end{aligned}$$

  so that the total equation of state $w \equiv p/\rho < -1/3$ for acceleration
- Conservation equation $\nabla^\mu T_{\mu\nu} = 0$ implies

$$\frac{\dot\rho}{\rho} = -3(1+w)\frac{\dot a}{a}$$

  so that ρ must scale more slowly than a^{-2}

w is not true Equation of State!

- But $w = p/\rho$ cannot be a functional relationship $p(\rho)$ for matter that accelerates the expansion
- For perturbations, such a true equation of state

$$\delta p = \frac{\partial p}{\partial \rho} \delta \rho$$

and for $p(\rho)$ would imply

$$dp/dt = (dp/d\rho)(d\rho/dt) \to dp/d\rho = \dot{p}/\dot{\rho}$$

- For example if $w =$ const. then $\delta p = -w\delta\rho$, and for $w < 0$, a negative pressure response to a positive density fluctuation so pressure gradients would make overdensities collapse rather than stabilize them in hydro equilibrium with gravity

w : EOS Param of Background

- Instead $w = p/\rho$ is a parameter that expresses a symmetry statement about the stress-energy tensor background pressure and energy density
- They are the only components consistent with homogeneity and isotropy by definition but apply to the spatially averaged dynamics, i.e. Friedmann equations not local evolution of matter-energy
- A dark energy candidate beyond Λ must introduce additional freedom $p(\rho, \ldots)$ to set the sound speed $dp/d\rho$ and avoid this (pressure) gradient instability
- Generalize this consideration beyond GR in the EFT of dark energy, where there is a second condition for ghost instability (wrong sign kinetic term, quantum instability)
- Can also have dissipative dark energy terms in δp (bulk viscosity) and π (scalar shear viscosity) or source GW (tensor π)

Scalar, Vector, Tensor

- In linear perturbation theory, perturbations may be separated by their transformation properties under 3D rotation and translation
- The eigenfunctions of the Laplacian operator form a complete set

$$
\begin{aligned}
\nabla^2 Q^{(0)} &= -k^2 Q^{(0)} \qquad && \mathrm{S}\,, \\
\nabla^2 Q_i^{(\pm 1)} &= -k^2 Q_i^{(\pm 1)} \qquad && \mathrm{V}\,, \\
\nabla^2 Q_{ij}^{(\pm 2)} &= -k^2 Q_{ij}^{(\pm 2)} \qquad && \mathrm{T}\,,
\end{aligned}
$$

- Vector and tensor modes satisfy divergence-free and transverse-traceless conditions

$$
\begin{aligned}
\nabla^i Q_i^{(\pm 1)} &= 0 \\
\nabla^i Q_{ij}^{(\pm 2)} &= 0 \\
\gamma^{ij} Q_{ij}^{(\pm 2)} &= 0
\end{aligned}
$$

Vector and Tensor Quantities

- A scalar mode carries with it associated vector (curl-free) and tensor (longitudinal) quantities
- A vector mode carries and associated tensor (trace and divergence free) quantities
- A tensor mode has only a tensor (trace and divergence free)
- These are built from the mode basis out of covariant derivatives and the metric

$$\begin{aligned}
Q_i^{(0)} &= -k^{-1}\nabla_i Q^{(0)} \,, \\
Q_{ij}^{(0)} &= (k^{-2}\nabla_i\nabla_j + \frac{1}{3}\gamma_{ij})Q^{(0)} \,, \\
Q_{ij}^{(\pm 1)} &= -\frac{1}{2k}[\nabla_i Q_j^{(\pm 1)} + \nabla_j Q_i^{(\pm 1)}] \,,
\end{aligned}$$

Perturbation k-Modes

- For the kth eigenmode, the scalar components become

$$\begin{aligned} A(\mathbf{x}) &= A(k)\, Q^{(0)}\,, & H_L(\mathbf{x}) &= H_L(k)\, Q^{(0)}\,, \\ \delta\rho(\mathbf{x}) &= \delta\rho(k)\, Q^{(0)}\,, & \delta p(\mathbf{x}) &= \delta p(k)\, Q^{(0)}\,, \end{aligned}$$

  the vectors components become

$$B_i(\mathbf{x}) = \sum_{m=-1}^{1} B^{(m)}(k)\, Q_i^{(m)}\,, \qquad v_i(\mathbf{x}) = \sum_{m=-1}^{1} v^{(m)}(k)\, Q_i^{(m)}\,,$$

  and the tensors components

$$H_{Tij}(\mathbf{x}) = \sum_{m=-2}^{2} H_T^{(m)}(k)\, Q_{ij}^{(m)}\,, \quad \Pi_{ij}(\mathbf{x}) = \sum_{m=-2}^{2} \Pi^{(m)}(k)\, Q_{ij}^{(m)}\,,$$

- Note that the curvature perturbation only involves scalars

$$\delta[{}^{(3)}R] = \frac{4}{a^2}(k^2 - 3K)(H_L^{(0)} + \frac{1}{3}H_T^{(0)})Q^{(0)}$$

Spatially Flat Case

- For a spatially flat background metric, harmonics are related to plane waves:

$$
\begin{aligned}
Q^{(0)} &= \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_i^{(\pm1)} &= \frac{-i}{\sqrt{2}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i \exp(i\mathbf{k}\cdot\mathbf{x}) \\
Q_{ij}^{(\pm2)} &= -\sqrt{\frac{3}{8}}(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_i(\hat{\mathbf{e}}_1 \pm i\hat{\mathbf{e}}_2)_j \exp(i\mathbf{k}\cdot\mathbf{x})
\end{aligned}
$$

  where $\hat{\mathbf{e}}_3 \parallel \mathbf{k}$. Chosen as spin states, cf. polarization.

- For vectors, the harmonic points in a direction orthogonal to $\mathbf{k}$ suitable for the vortical component of a vector

- Under rotations around $\mathbf{e}_3 = \hat{\mathbf{k}}$, these modes transform into themselves and just acquire a phase

Spatially Flat Case

- Tensor harmonics are the transverse traceless gauge representation
- Tensor amplitude related to the more traditional

$$h_+[(\mathbf{e}_1)_i(\mathbf{e}_1)_j - (\mathbf{e}_2)_i(\mathbf{e}_2)_j]\,, \qquad h_\times[(\mathbf{e}_1)_i(\mathbf{e}_2)_j + (\mathbf{e}_2)_i(\mathbf{e}_1)_j]$$

  as

$$h_+ \pm i h_\times = -\sqrt{6} H_T^{(\mp 2)}$$

- $H_T^{(\pm 2)}$ proportional to the right and left circularly polarized amplitudes of gravitational waves with a normalization that is convenient to match the scalar and vector definitions

Covariant Scalar Equations

- DOF counting exercise

8	Variables (4 metric; 4 matter)
-4	Einstein equations
-2	Conservation equations
$+2$	Bianchi identities
-2	Gauge (coordinate choice 1 time, 1 space)
2	Free Variables

without loss of generality choose scalar components of the stress tensor $\delta p, \Pi$.

Covariant Scalar Equations

- Einstein equations (suppressing 0) superscripts

$$(k^2 - 3K)[H_L + \frac{1}{3}H_T] - 3(\frac{\dot a}{a})^2 A + 3\frac{\dot a}{a}\dot H_L + \frac{\dot a}{a}kB =$$

$$= 4\pi G a^2 \delta\rho\,, \qquad \text{00 Poisson Equation}$$

$$\frac{\dot a}{a}A - \dot H_L - \frac{1}{3}\dot H_T - \frac{K}{k^2}(kB - \dot H_T)$$

$$= 4\pi G a^2 (\rho + p)(v - B)/k\,, \qquad 0i \text{ Momentum Equation}$$

$$\left[2\frac{\ddot a}{a} - 2\left(\frac{\dot a}{a}\right)^2 + \frac{\dot a}{a}\frac{d}{d\eta} - \frac{k^2}{3}\right] A - \left[\frac{d}{d\eta} + \frac{\dot a}{a}\right](\dot H_L + \frac{1}{3}kB)$$

$$= 4\pi G a^2 (\delta p + \frac{1}{3}\delta\rho)\,, \qquad ii \text{ Acceleration Equation}$$

$$k^2(A + H_L + \frac{1}{3}H_T) + \left(\frac{d}{d\eta} + 2\frac{\dot a}{a}\right)(kB - \dot H_T)$$

$$= -8\pi G a^2 p \Pi\,, \qquad ij \text{ Anisotropy Equation}$$

Covariant Scalar Equations

- Poisson and acceleration equations are the perturbed generalization of the Friedmann equations
- Momentum and anisotropy equations are new to the perturbed metric
- Poisson and momentum equations in the ADM language take the form of constraints on the shift and lapse respectively - leaving the spatial metric components as dynamical
- Like the Friedmann equations, the 4 equation are redundant given the 2 energy-momentum conservation equations
- Choose a gauge and set of equations to simplify the given problem

Covariant Scalar Equations

- Conservation equations: continuity and Navier Stokes

$$\begin{aligned}\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho + 3\frac{\dot a}{a}\delta p &= -(\rho+p)(kv + 3\dot H_L)\,, \\ \left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right]\left[(\rho+p)\frac{(v-B)}{k}\right] &= \delta p - \frac{2}{3}(1-3\frac{K}{k^2})p\Pi + (\rho+p)A\,,\end{aligned}$$

- Equations are not independent since $\nabla_\mu G^{\mu\nu} = 0$ via the Bianchi identities.

- Related to the ability to choose a coordinate system or “gauge” to represent the perturbations.

Covariant Vector Equations

- Einstein equations

$$\begin{aligned}(1 - 2K/k^2)(kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) \\ = 16\pi G a^2 (\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k\,, \\ \left[\frac{d}{d\eta} + 2\frac{\dot{a}}{a}\right](kB^{(\pm 1)} - \dot{H}_T^{(\pm 1)}) \\ = -8\pi G a^2 p \Pi^{(\pm 1)}\,.\end{aligned}$$

- Conservation Equations

$$\begin{aligned}\left[\frac{d}{d\eta} + 4\frac{\dot{a}}{a}\right][(\rho + p)(v^{(\pm 1)} - B^{(\pm 1)})/k] \\ = -\frac{1}{2}(1 - 2K/k^2) p \Pi^{(\pm 1)}\,,\end{aligned}$$

- Gravity provides no source to vorticity → decay

Covariant Vector Equations

- DOF counting exercise

8	Variables (4 metric; 4 matter)
-4	Einstein equations
-2	Conservation equations
$+2$	Bianchi identities
-2	Gauge (coordinate choice 1 time, 1 space)
2	Free Variables

without loss of generality choose vector components of the stress tensor $\Pi^{(\pm1)}$.

Covariant Tensor Equation

- Einstein equation

$$\left[\frac{d^2}{d\eta^2} + 2\frac{\dot{a}}{a}\frac{d}{d\eta} + (k^2 + 2K)\right] H_T^{(\pm 2)} = 8\pi G a^2 p \Pi^{(\pm 2)} \,.$$

- DOF counting exercise

4	Variables (2 metric; 2 matter)
-2	Einstein equations
-0	Conservation equations
$+0$	Bianchi identities
-0	Gauge (coordinate choice 1 time, 1 space)
2	Free Variables

wlog choose tensor components of the stress tensor $\Pi^{(\pm 2)}$.

Arbitrary Dark Components

- Total stress energy tensor can be broken up into individual pieces
- Dark components interact only through gravity and so satisfy separate conservation equations
- Einstein equation source remains the sum of components.
- To specify an arbitrary dark component, give the behavior of the stress tensor: 6 components: δp, $\Pi^{(i)}$, where $i = -2, ..., 2$.
- Many types of dark components (dark matter, scalar fields, massive neutrinos,..) have simple forms for their stress tensor in terms of the energy density, i.e. described by equations of state but w for the background is *not* sufficient
- Parameterized functions: "generalized" dark matter (sound speed, viscosity); "parameterized post Friedmann" (Jeans scale, shear) for dark energy

Geometry of Gauge Choice

- Geometry of the gauge or time slicing and spatial threading
- For perturbations larger than the horizon, a local observer should just see a different (separate) FRW universe
- Scalar equations should be equivalent to an appropriately remapped Friedmann equation
- ADM recap: unit normal vector n^μ to constant time hypersurfaces $n_\mu dx^\mu = n_0 d\eta$, $n^\mu n_\mu = -1$, to linear order in metric

$$\begin{aligned} n_0 &= -a(1+AQ), \qquad n_i = 0 \\ n^0 &= a^{-1}(1-AQ), \qquad n^i = -BQ^i \end{aligned}$$

- Intrinsic 3-geometry of δg_{ij}, changes in the normal vector $n_{\mu;\nu}$ that define the extrinsic curvature

Geometric Quantities

- Expansion of spatial volume per proper time is given by 4-divergence

$$n^{\mu}{}_{;\mu} \equiv \theta = 3H(1 - AQ) + \frac{k}{a}BQ + \frac{3}{a}\dot{H}_L Q$$

- Other pieces of $n_{\mu;\nu}$ give the vorticity, shear and acceleration

$$n_{\mu;\nu} \equiv \omega_{\mu\nu} + \sigma_{\mu\nu} + \frac{\theta}{3}h_{\mu\nu} - a_\mu n_\nu$$

$$\begin{aligned} h_{\mu\nu} &= g_{\mu\nu} + n_\mu n_\nu \\ \omega_{\mu\nu} &= h_\mu{}^\alpha h_\nu{}^\beta (n_{\alpha;\beta} - n_{\beta;\alpha}) = 0 \\ \sigma_{\mu\nu} &= \frac{1}{2}h_\mu{}^\alpha h_\nu{}^\beta (n_{\alpha;\beta} + n_{\beta;\alpha}) - \frac{1}{3}\theta h_{\mu\nu} \\ a_\mu &= n_{\mu;\alpha}n^\alpha \end{aligned}$$

- Recall n_μ is a special timelike vector normal to the constant time surfaces, the vorticity vanishes by construction

Geometric Quantities

- Remaining perturbed quantities are the spatial shear and acceleration (0 components vanish)

$$\begin{aligned} \sigma_{ij} &= a(\dot{H}_T - kB)Q_{ij} \\ a_i &= -kAQ_i \end{aligned}$$

- Recall that the extrinsic curvature $K_{ij} = \sigma_{ij} + \theta h_{ij}/3$
- Intrinsic curvature of the 3-surface determined by 3-metric h_{ij}

$$\delta[^{(3)}R] = \frac{4}{a^2}(k^2 - 3K)(H_L + \frac{H_T}{3})$$

- E-foldings of the local expansion $\ln a_L$ are given

$$\ln a_L = \int d\tau \frac{1}{3}\theta = \int d\eta \left(\frac{\dot{a}}{a} + \dot{H}_L Q + \frac{1}{3} kBQ \right)$$

where we have used $d\tau = (1 + AQ)ad\eta$

Separate Universe

- Notice that

$$\frac{d}{d\eta}\delta \ln a_L = \dot{H}_L + \frac{\dot{H}_T}{3} - \frac{1}{3}(\dot{H}_T - kB)$$

  so that if the shear is negligible the change in efolds tracks the change in curvature

- Shear vanishes in the FRW background; uniform efolding gives constant curvature
- Underlying principle: local observer should find long wavelength perturbations are indistingishable from a change in the background FRW quantities
- Perturbation equations take the form of Friedmann equations once rescaled

Time Slicing

- Constant time surfaces can be defined according to what geometry is helpful for the problem at hand
- Common choices:
  - Shear free: $\dot{H}_T - kB = 0$
  - Zero lapse pert or acceleration, $A = 0$
  - Uniform density: $\delta\rho = 0$
  - Spatially flat (unperturbed ${}^{(3)}R$): $H_L + H_T/3 = 0$
  - Comoving: $v = B$ ($n^\mu \propto V^\mu$, see below)
  - Uniform expansion: $-3HA + (3\dot{H}_L + kB) = 0$
  - Uniform efolding: $\dot{H}_L + kB/3 = 0$
- For the background all of these conditions hold.
- For perturbations each define a choice of slicing
- Can define the validity of the separate universe principle as the coexistence of comoving and zero lapse slicing

Time Slicing

- Comoving slicing is more formally called velocity orthogonal slicing since constant time surfaces are orthogonal to the matter 4-velocity V^μ:

$$h^\mu{}_\nu V^\nu = (\delta^\mu{}_\nu + n^\mu n_\nu)V^\nu = (0, V^i + N^i V^0) = 0$$

$$\rightarrow V^i = vQ^i = B^i = BQ^i$$

- Should not be confused with comoving (threading) where the 3-velocity $v = 0$ unless the shift B also vanishes
- Useful to have finite $B = v$ so that matter moves on the spatial slice, so that coordinates are more "Eulerian" than"Lagrangian"
- Q: how do we know what are valid conditions on slicing? check whether it can be established by the arbitrary coordinate, or gauge choice

Gauge

- Metric and matter fluctuations take on different values in different coordinate system
- No such thing as a “gauge invariant” density perturbation!
- General coordinate transformation:

  $$\begin{aligned} \tilde{\eta} &= \eta + T \\ \tilde{x}^i &= x^i + L^i \end{aligned}$$

  free to choose (T, L^i) to simplify equations or physics – corresponds to a choice of slicing and threading in ADM.
- Decompose these into scalar T, $L^{(0)}$ and vector harmonics $L^{(\pm 1)}$.

Gauge

- $g_{\mu\nu}$ and $T_{\mu\nu}$ transform as tensors, so components in different frames can be related

$$\begin{aligned}\tilde{g}_{\mu\nu}(\tilde{\eta},\tilde{x}^i) &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\eta, x^i) \\ &= \frac{\partial x^\alpha}{\partial \tilde{x}^\mu}\frac{\partial x^\beta}{\partial \tilde{x}^\nu} g_{\alpha\beta}(\tilde{\eta} - TQ, \tilde{x}^i - LQ^i)\end{aligned}$$

- Fluctuations are compared at the same coordinate positions (not same space time positions) between the two gauges
- For example with a TQ perturbation, an event labeled with $\tilde{\eta}$ =const. and $\tilde{x}$ =const. represents a different time with respect to the underlying homogeneous and isotropic background

Vector Gauges

- Vector gauge depends only on threading L
- Poisson gauge: orthogonal threading $B^{(\pm 1)} = 0$, leaves constant L translational freedom
- Isotropic gauge: isotropic threading $H_T^{(\pm 1)} = 0$, fixes L
- To first order scalar and vector gauge conditions can be chosen separately
- More care required for second and higher order where scalars and vectors mix

Gauge Transformation

- Scalar Metric:

$$\begin{aligned}
\tilde{A} &= A - \dot{T} - \frac{\dot{a}}{a}T\,, \\
\tilde{B} &= B + \dot{L} + kT\,, \\
\tilde{H}_L &= H_L - \frac{k}{3}L - \frac{\dot{a}}{a}T\,, \\
\tilde{H}_T &= H_T + kL\,, \qquad \tilde{H}_L + \frac{1}{3}\tilde{H}_T = H_L + \frac{1}{3}H_T - \frac{\dot{a}}{a}T
\end{aligned}$$

curvature perturbation depends on slicing not threading

- Scalar Matter (Jth component):

$$\begin{aligned}
\delta\tilde{\rho}_J &= \delta\rho_J - \dot{\rho}_J T\,, \\
\delta\tilde{p}_J &= \delta p_J - \dot{p}_J T\,, \\
\tilde{v}_J &= v_J + \dot{L}\,,
\end{aligned}$$

density and pressure likewise depend on slicing only

Gauge Transformation

- Vector:

$$\begin{aligned}
\tilde{B}^{(\pm1)} &= B^{(\pm1)} + \dot{L}^{(\pm1)}, \\
\tilde{H}_T^{(\pm1)} &= H_T^{(\pm1)} + kL^{(\pm1)}, \\
\tilde{v}_J^{(\pm1)} &= v_J^{(\pm1)} + \dot{L}^{(\pm1)},
\end{aligned}$$

- Spatial vector has no background component hence no dependence on slicing at first order

  Tensor: no dependence on slicing or threading at first order

- Gauge transformations and covariant representation can be extended to higher orders

- A coordinate system is fully specified if there is an explicit prescription for (T, L^i) or for scalars (T, L)

Slicing

Common choices for slicing T: set something geometric to zero

- Proper time slicing $A = 0$: proper time between slices corresponds to coordinate time – T allows c/a freedom
- Comoving (velocity orthogonal) slicing: $v - B = 0$, slicing is orthogonal to matter 4 velocity - T fixed

  Can be specialized to velocity orthogonal to a specific, e.g. dark energy, component - define "gauge invariant" sound speed
- Newtonian (shear free) slicing: $\dot{H}_T - kB = 0$, expansion rate is isotropic, shear free, T fixed
- Uniform expansion slicing: $-(\dot{a}/a)A + \dot{H}_L + kB/3 = 0$, perturbation to the volume expansion rate θ vanishes, T fixed
- Flat (constant curvature) slicing, $\delta^{(3)}R = 0$, $(H_L + H_T/3 = 0)$, T fixed
- Constant density slicing, $\delta\rho_I = 0$, T fixed

Scalar Field Example: Inflaton, Axion...

- A canonical scalar field can drive inflation by supplying potential energy that doesn't change at a fixed field value as the universe expands

$$S_\phi = \int d^4x\sqrt{-g}[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]$$

- Varying the action with respect to the metric

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}\mathcal{L}_\phi$$

  gives the stress-energy tensor of a scalar field

$$T^\mu{}_\nu = \nabla^\mu\phi\,\nabla_\nu\phi - \frac{1}{2}(\nabla^\alpha\phi\,\nabla_\alpha\phi + 2V)\delta^\mu{}_\nu\,.$$

- Equations of motion $\nabla^\mu T_{\mu\nu} = 0$ with closure relations for $p(\phi, \partial_\mu\phi)$, $\Pi(\phi, \partial_\mu\phi)$ or field equation $\nabla_\mu\nabla^\mu\phi = V'$ (vary with respect to ϕ)

Scalar Fields

- For the background $\langle\phi\rangle \equiv \phi(\eta)$ (a^{-2} from conformal time)

$$\rho_\phi = \frac{1}{2}a^{-2}\dot{\phi}^2 + V, \quad p_\phi = \frac{1}{2}a^{-2}\dot{\phi}^2 - V$$

- So for kinetic dominated $w_\phi = p_\phi/\rho_\phi \to 1$
- And potential dominated $w_\phi = p_\phi/\rho_\phi \to -1$
- Can use general equations of motion of dictated by stress energy conservation

$$\dot{\rho}_\phi = -3(\rho_\phi + p_\phi)\frac{\dot{a}}{a}\,,$$

to obtain the equation of motion of the background field ϕ

$$\ddot{\phi} + 2\frac{\dot{a}}{a}\dot{\phi} + a^2 V' = 0\,,$$

$$\frac{d^2\phi}{dt^2} + 3H\frac{d\phi}{dt} + V' = 0$$

Field Perturbations

- Perturbations of the field $\phi = \phi + \delta\phi$ evolve under the continuity and Euler equations

$$
\begin{aligned}
\delta\rho_\phi &= a^{-2}(\dot\phi\dot{\delta\phi} - \dot\phi^2 A) + V'\delta\phi\,, \\
\delta p_\phi &= a^{-2}(\dot\phi\dot{\delta\phi} - \dot\phi^2 A) - V'\delta\phi\,, \\
(\rho_\phi + p_\phi)(v_\phi - B) &= a^{-2}k\dot\phi\delta\phi\,, \\
p_\phi\pi_\phi &= 0\,,
\end{aligned}
$$

- For comoving slicing during inflation or ϕ-comoving ($v_\phi = B$) for any FRW background

$$\delta\phi = 0 \quad \text{unperturbed}$$

- Nonetheless $\delta\rho_\phi \neq 0$ and the "comoving" or rest frame sound speed is $c_s^2 \equiv \delta p_\phi/\delta\rho_\phi = 1$

Threading

- Threading specifies the relationship between constant spatial coordinates between slices and is determined by L

  Typically involves a condition on v, B, H_T
  - Orthogonal threading $B = 0$, constant spatial coordinates orthogonal to slicing (zero shift), allows $\delta L = c$ translational freedom
  - Comoving threading $v = 0$, allows $\delta L = c$ translational freedom.
  - Isotropic threading $H_T = 0$, fully fixes L

Newtonian (Longitudinal) Gauge

- Newtonian (shear free slicing, isotropic threading):

$$\begin{aligned} \tilde{B} &= \tilde{H}_T = 0 \\ \Psi &\equiv \tilde{A} \quad \text{(Newtonian potential)} \\ \Phi &\equiv \tilde{H}_L \quad \text{(Newtonian curvature)} \\ L &= -H_T/k \\ T &= -B/k + \dot{H}_T/k^2 \end{aligned}$$

Good: intuitive Newtonian like gravity; matter and metric algebraically related; commonly chosen for analytic CMB and lensing work

Bad: numerically unstable for superhorizon fluctuations, but can easily be remedied in the "hybrid approach" by chosing dynamical linear combinations

Newtonian (Longitudinal) Gauge

- Newtonian (shear free) slicing, isotropic threading $B = H_T = 0$:

$$\begin{aligned}(k^2 - 3K)\Phi &= 4\pi G a^2 \left[\delta\rho + 3\frac{\dot a}{a}(\rho+p)v/k\right] \quad \text{Poisson + Momentum} \\ k^2(\Psi+\Phi) &= -8\pi G a^2 p\Pi \quad \text{Anisotropy}\end{aligned}$$

so $\Psi = -\Phi$ if anisotropic stress $\Pi = 0$ and

$$\begin{aligned}\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho + 3\frac{\dot a}{a}\delta p &= -(\rho+p)(kv + 3\dot\Phi)\,, \\ \left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho+p)v &= k\delta p - \frac{2}{3}(1 - 3\frac{K}{k^2})p\,k\Pi + (\rho+p)\,k\Psi\,,\end{aligned}$$

- Newtonian competition between stress (pressure and viscosity) and potential gradients
- Note: Poisson source is the density perturbation on comoving slicing

Comoving Gauge

- Comoving gauge (comoving slicing, isotropic threading)

$$\begin{aligned}
\tilde{v} - \tilde{B} &= 0 \quad (T^0_i = 0) \\
\tilde{H}_T &= 0 \\
\xi &\equiv \tilde{A} \\
\mathcal{R} &\equiv \tilde{H}_L \quad \text{(comoving curvature)} \\
\Delta &\equiv \tilde{\delta} \quad \text{(total density pert)} \\
T &= (v - B)/k \\
L &= -H_T/k
\end{aligned}$$

Good: Algebraic relations between matter and metric; comoving curvature perturbation obeys conservation law

Bad (but useful): Non-intuitive: slicing is comoving but threading is not due to shift condition

Comoving Gauge

- Euler equation becomes an algebraic relation between stress and potential

$$(\rho + p)\xi = -\delta p + \frac{2}{3}\left(1 - \frac{3K}{k^2}\right) p\Pi$$

- Einstein equation lacks momentum density source

$$\frac{\dot{a}}{a}\xi - \dot{\mathcal{R}} - \frac{K}{k^2}kv = 0$$

Combine: $\mathcal{R}$ is conserved if stress fluctuations negligible, e.g. above the horizon if $|K| \ll H^2$

$$\dot{\mathcal{R}} + Kv/k = \frac{\dot{a}}{a}\left[-\frac{\delta p}{\rho + p} + \frac{2}{3}\left(1 - \frac{3K}{k^2}\right)\frac{p}{\rho + p}\Pi\right] \to 0$$

Sound Speed

- In this slicing the perturbation to the energy density and pressure due to a change in field position on the potential is absent, so they reflect the kinetic piece while the background is dominated by the potential piece

  $$\delta p_\phi = \delta \rho_\phi$$

  so the sound speed is $\delta p_\phi / \delta \rho_\phi = 1$.

- More generally the sound speed of the inflation is defined as the speed at which field fluctuations propagate - i.e. the kinetic piece to the energy density rather than the $V' \delta\phi$ potential piece - much like in the background the $+1$ and -1 pieces of w.

- Non canonical kinetic terms EFT– k-essence, DBI inflation, Horndeski – can generate $c_s \neq 1$ as do terms in the effective theory of inflation

“Gauge Invariant” Approach

- Gauge transformation rules allow variables which take on a geometric meaning in one choice of slicing and threading to be accessed from variables on another choice
- Functional form of the relationship between the variables is gauge invariant (*not* the variable values themselves! – i.e. equation is *covariant*)
- E.g. comoving curvature and density perturbations

$$\begin{aligned} \mathcal{R} &= H_L + \frac{1}{3}H_T - \frac{\dot a}{a}(v - B)/k \\ \Delta\rho &= \delta\rho + 3(\rho + p)\frac{\dot a}{a}(v - B)/k \end{aligned}$$

Newtonian-Comoving Hybrid

- With the gauge in(*or co*)variant approach, express variables of one gauge in terms of those in another – allows a mixture in the equations of motion
- Example: Newtonian curvature and comoving density

  $$(k^2 - 3K)\Phi = 4\pi G a^2 \rho \Delta$$

  ordinary Poisson equation then implies Φ approximately constant if stresses negligible.
- Example: Exact Newtonian curvature above the horizon derived through comoving curvature conservation

  Gauge transformation

  $$\Phi = \mathcal{R} + \frac{\dot a}{a}\frac{v}{k}$$

Hybrid “Gauge Invariant” Approach

Einstein equation to eliminate velocity

$$\frac{\dot{a}}{a}\Psi - \dot{\Phi} = 4\pi G a^2(\rho + p)v/k$$

Friedmann equation with no spatial curvature

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}a^2\rho$$

With $\dot{\Phi} = 0$ and $\Psi \approx -\Phi$

$$\frac{\dot{a}}{a}\frac{v}{k} = -\frac{2}{3(1+w)}\Phi$$

Newtonian-Comoving Hybrid

Combining gauge transformation with velocity relation

$$\Phi = \frac{3+3w}{5+3w}\mathcal{R}$$

Usage: calculate $\mathcal{R}$ from inflation determines Φ for any choice of matter content or causal evolution.

- Example: Scalar field ("quintessence" dark energy) equations in comoving gauge imply a sound speed $\delta p/\delta\rho = 1$ independent of potential $V(\phi)$. Solve in synchronous gauge.
- More generally, components can often be modeled as an imperfect fluid with non-adiabatic stress and viscosity...

Arbitrary Dark Component Redux

- Close the conservation equations for the Jth component with isotropic stress $\delta p_J(\delta\rho_J, v_J, ...)$ and anisotropic stress π_J
- Go to slicing that is comoving with respect to J and the expansion is isotropic $H_T = 0$ and call that the fluid rest gauge $v^r_J = B^r$
- Model as an imperfect fluid: anisotropic stress π_J if there is shear viscosity: isotropic stress with a sound speed, bulk viscosity c^2_{bv}

$$\delta p^r_J = c^2_s \delta\rho^r_J + c^2_{bv}(\rho_J + p_J)\frac{kv - \dot{H}_T}{aH} \tag{1}$$

  viscous term is gauge invariant (same value in all gauges)

- Arbitrary gauge: isotropic stress δp is modeled covariantly as

$$\delta p = c^2_s \delta\rho + 3(c^2_s - c^2_a)(1+w)\rho\frac{\dot a}{a}\frac{v - B}{k} + c^2_{bv}(\rho + p)\frac{kv - \dot{H}_T}{aH} \tag{2}$$

  where $c^2_a = \dot p/\dot\rho$ is the adiabatic sound speed - nonadiabatic stress if $c^2_s \neq c^2_a$ or $c^2_{bv} \neq 0$. Can generalize further to shear viscosity

Sachs-Wolfe Effect

- On superhorizon scales $k\eta \ll 1$, $\Delta\rho/\rho \ll \mathcal{R}$ and $\xi \ll \mathcal{R}$ in comoving gauge for adiabatic perturbations $p(\rho)$
- Both $\mathcal{R}$ and Ψ, Φ are constant when $w =$const.
- Derive the observed CMB temperature fluctuation for superhorizon fluctuations at recombination: Sachs-Wolfe effect
- Time shift from Newtonian lapse Ψ to comoving lapse

$$\xi = \Psi - \dot{T} - \frac{\dot{a}}{a}T$$

- For $k\eta \ll 1$, $\Delta \ll \mathcal{R} = O(\Psi)$ and $\Delta p/(\rho + p) = O(\Delta)$ so $\xi \ll \mathcal{R} \sim |\Psi|$

$$\dot{T} + \frac{\dot{a}}{a}T \approx \Psi \rightarrow T \approx a^{-1}\int a\Psi d\eta \approx a^{-1}\Psi t$$

Sachs-Wolfe Effect

- Time shift induces a density perturbation $\delta t = a\delta\eta = aT$ so $\delta t/t = \Psi$, and $t = \int d\ln a/H \propto a^{3(1+w)/2}$

$$\frac{\delta t}{t} = \frac{3}{2}(1+w)\frac{\delta a}{a} = \Psi$$

- $T_{\rm CMB} \propto a^{-1}$ so

$$\frac{\delta T_{\rm CMB}}{T_{\rm CMB}}\bigg|_{\rm local} = -\frac{\delta a}{a} = -\frac{2}{3(1+w)}\Psi$$

- Correct for gravitational redshift from climbing out of Ψ

$$\frac{\delta T_{\rm CMB}}{T_{\rm CMB}}\bigg|_{\rm obs} = \frac{\delta T_{\rm CMB}}{T_{\rm CMB}}\bigg|_{\rm local} + \Psi = \frac{1+3w}{3(1+w)\Psi}$$

- So in matter domination $\Psi/3 = \mathcal{R}/5$ and in radiation domination $\Psi/2 = \mathcal{R}/3$

Sachs-Wolfe Effect

- So measurement of $\delta T_{\rm CMB}/T_{\rm CMB} \approx 10^{-5}$ at largest angles implies the initial comoving curvature $\mathcal{R} \approx 5 \times 10^{-5}$ or $|\mathcal{R}|^2 = A_s \approx 2.5 \times 10^{-9}$
- Small red tilt of the spectrum and modern normalization point of $k_0 = 0.05{\rm Mpc}^{-1}$ gives a reduction in the Planck measured value of A_s

$$\frac{k^3 P_{\mathcal{R}}}{2\pi^2} = A_s \left(\frac{k}{k_0}\right)^{n_s - 1}$$

- Reversing the argument: measured large scale anisotropy implies a curvature fluctuation above the horizon - since curvature is conserved outside the horizon this comes from a period of acceleration in the early universe where fluctuations were inside the horizon

Synchronous Gauge

- Synchronous: (proper time slicing, orthogonal threading)

$$\begin{aligned}
\tilde{A} &= \tilde{B} = 0 \\
\eta_T &\equiv -\tilde{H}_L - \frac{1}{3}\tilde{H}_T \\
h_L &\equiv 6H_L \\
T &= a^{-1}\int d\eta a A + c_1 a^{-1} \\
L &= -\int d\eta (B + kT) + c_2
\end{aligned}$$

Good: stable, the choice of numerical codes and separate universe constructs

Bad: residual gauge freedom in constants c_1, c_2 must be specified as an initial condition, intrinsically relativistic, threading conditions breaks down beyond linear regime if c_1 is fixed to CDM rest frame

Synchronous Gauge

- Residual gauge freedom in time slicing: multiple synchronous gauges related by

$$T = \frac{c_1}{a}$$

- Notice that momentum transforms with

$$\tilde{v} - \tilde{B} = v - B - kT \rightarrow \tilde{v} = v - \frac{kc_1}{a}$$

- An initial velocity in the absence of gravitational and pressure decays with expansion as $v \propto 1/a$
- Time slicing freedom is associated with the initial velocity of synchronous observers - set this to zero - via CDM as observers
- Spatial residual freedom c_2 associated with the spatial grid of synchronous observers - usually set this to be uniform in comoving coordinates - via CDM as observers

Synchronous Gauge

- The Einstein equations give

$$\dot\eta_T - \frac{K}{2k^2}(\dot h_L + 6\dot\eta_T) = 4\pi G a^2(\rho + p)\frac{v}{k}\,,$$

$$\ddot h_L + \frac{\dot a}{a}\dot h_L = -8\pi G a^2(\delta\rho + 3\delta p)\,,$$

$$-(k^2 - 3K)\eta_T + \frac{1}{2}\frac{\dot a}{a}\dot h_L = 4\pi G a^2\delta\rho$$

[choose (1 & 2) or (1 & 3)] while the conservation equations give

$$\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho_J + 3\frac{\dot a}{a}\delta p_J = -(\rho_J + p_J)(kv_J + \frac{1}{2}\dot h_L)\,,$$

$$\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho_J + p_J)\frac{v_J}{k} = \delta p_J - \frac{2}{3}(1 - 3\frac{K}{k^2})p_J\Pi_J\,.$$

Synchronous Gauge

- Lack of a lapse A implies no gravitational forces in Navier-Stokes equation. Hence for stress free matter like cold dark matter, zero velocity initially implies zero velocity always.
- Choosing the momentum and acceleration Einstein equations is good since for CDM domination, curvature η_T is conserved and $\dot{h}_L$ is simple to solve for.
- Choosing the momentum and Poisson equations is good when the equation of state of the matter is complicated since δp is not involved. This is the choice of CAMB.

  Caution: since the curvature η_T appears and it has zero CDM source, subtle effects like dark energy perturbations are important everywhere

Spatially Flat Gauge

- Spatially Flat (flat slicing, isotropic threading):

$$\begin{aligned}
\tilde{H}_L &= \tilde{H}_T = 0 \\
L &= -H_T/k \\
\tilde{A}\,, \tilde{B} &= \quad \text{metric perturbations} \\
T &= \left(\frac{\dot{a}}{a}\right)^{-1} \left(H_L + \frac{1}{3} H_T\right)
\end{aligned}$$

Good: eliminates spatial metric evolution in ADM and perturbation equations ; useful in inflationary calculations (Mukhanov et al)

Bad: eliminates curvature so requires a gauge transformation or δN-efold technique to link to structure formation

- Caution: perturbation evolution is governed by the behavior of stress fluctuations and an isotropic stress fluctuation δp is gauge dependent.

Uniform Density Gauge

- Uniform density: (constant density slicing, isotropic threading)

$$
\begin{aligned}
H_T &= 0\,, \\
\zeta_I &\equiv H_L \\
B_I &\equiv B \\
A_I &\equiv A \\
T &= \frac{\delta\rho_I}{\dot\rho_I} \\
L &= -H_T/k
\end{aligned}
$$

Good: Curvature conserved involves only stress energy conservation; simplifies isocurvature treatment
Bad: non intuitive slicing (no density pert! problems beyond linear regime) and threading

Uniform Density Gauge

- Einstein equations simplify if I as the total or dominant species

$$(k^2 - 3K)\zeta_I - 3\left(\frac{\dot a}{a}\right)^2 A_I + 3\frac{\dot a}{a}\dot\zeta_I + \frac{\dot a}{a}kB_I = 0\,,$$

$$\frac{\dot a}{a}A_I - \dot\zeta_I - \frac{K}{k}B_I = 4\pi G a^2(\rho+p)\frac{v - B_I}{k}\,,$$

  More generally the Poisson source could involve other species J

- The conservation equations for a general component J (if $J = I$ then $\delta\rho_J = 0$)

$$\left[\frac{d}{d\eta} + 3\frac{\dot a}{a}\right]\delta\rho_J + 3\frac{\dot a}{a}\delta p_J = -(\rho_J + p_J)(kv_J + 3\dot\zeta_I)\,,$$

$$\left[\frac{d}{d\eta} + 4\frac{\dot a}{a}\right](\rho_J + p_J)\frac{v_J - B_I}{k} = \delta p_J - \frac{2}{3}(1 - 3\frac{K}{k^2})p_J\Pi_J + (\rho_J + p_J)A_I\,.$$

Uniform Density Gauge

- Conservation of curvature related to the stresses and velocity divergence of I

$$\dot{\zeta}_I = -\frac{\dot{a}}{a}\frac{\delta p_I}{\rho_I + p_I} - \frac{1}{3}kv_I \,.$$

- Since $\delta\rho_I = 0$, δp_I is the non-adiabatic stress and curvature is constant as $k \to 0$ for adiabatic fluctuations $p_I(\rho_I)$.

- Note that this conservation law does not involve the Einstein equations at all: just local energy momentum conservation so it is valid for alternate theories of gravity

- Curvature on comoving slices $\mathcal{R}$ and ζ_I related by

$$\zeta_I = \mathcal{R} + \frac{1}{3}\frac{\rho_I \Delta_I}{(\rho_I + p_I)}\Big|_{\rm comoving} \,.$$

  and coincide above the horizon for adiabatic fluctuations

Uniform Density Gauge

- Simple relationship to density fluctuations in the spatially flat gauge

$$\zeta_I = \frac{1}{3}\frac{\delta\tilde\rho_I}{(\rho_I + p_I)}\Big|_{\rm flat} .$$

- For each particle species $\delta\rho/(\rho+p) = \delta n/n$, the number density fluctuation
- Multiple ζ_J carry information about number density fluctuations between species
- ζ_J constant component by component outside horizon if each component is adiabatic $p_J(\rho_J)$.
- In cases where ζ_J is not constant due to internal non-adiabatic stress but the expansion shear is negligible, it can be computed by counting the efolds from a spatially flat hypersurface to a uniform density hypersurface: the δN approach for inflation

Unitary Gauge

- Given a scalar field $\phi(x^i, \eta)$, choose a slicing so that the field is spatially uniform $\phi(x^i, \eta) = \phi(\eta)$ via the transformation

$$\delta\tilde{\phi} = \delta\phi - \dot{\phi}T \quad \rightarrow \quad T = \frac{\delta\phi}{\dot{\phi}_0}$$

- Specify threading, e.g. isotropic threading $L = -H_T/k$

  Good: Scalar field carried completely by the metric; EFT of inflation and scalar-tensor theories of gravity. Extensible to nonlinear perturbations as long as $\partial_\mu\phi$ remains timelike

  Bad: Preferred slicing retains only the spatial diffeomorphism invariance; can make full covariance and DOF counting obscure

Unitary Gauge

- For a canonical scalar field as the *only* matter component (inflation), unitary and comoving gauge coincide, so keep notation as $\mathcal{R} = H_T + H_L/3$ and isotropic threading $H_T = 0$
- Generalized for dark energy, and will switch to $\zeta = H_T + H_L/3$, $H_T = 0$ to avoid confusion
- Following the general ADM/EFT construction, we revert to $x^0 = t$, i.e. ordinary coordinate time not conformal time and overdots $\cdot = \partial/\partial t$
- For notational simplicity, we will assume that the FRW curvature is negligible for the EFT treatment and take units of $M_{\rm pl} = 1$

EFT of Dark Energy and Inflation

- Beyond ΛCDM, unitary gauge and ADM is useful to define most general Lagrangian and interaction terms for a scalar-tensor theory of gravity: so-called Effective Field Theory (EFT)
- Rule: broken temporal diffeomorphisms (preferred slicing) but spatial diffeomorphism invariance means explicit functions of unitary time and ADM spatial scalars allowed

  $$\mathcal{L}(N, K^i{}_j, R^i{}_j, \ldots; t)$$

  where the function is constructed out of spatially covariant contractions (spatial scalars).
- Explicit time dependence can be considered a scalar field whose value is homogeneous $\phi(\mathbf{x}, t) = \phi(t) \rightarrow t(\phi)$
- Non-degenerate theories demand EOM second order in time derivatives to avoid Ostrogradsky ghost

EFT of Dark Energy and Inflation

- We can restore “gauge invariance” or time diff by promoting ϕ to a spacetime field with dynamics in the same way as going from unitary gauge to a general gauge
- Consider the dual language for a simple case where gravity is not modified and ϕ can be considered a dark component with its own dynamics: $\mathcal{L} = \mathcal{L}_\phi + \mathcal{L}_{\rm EH}$
- Simplest example k-essence where $\mathcal{L}_\phi(X,\phi)$ where $X = g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ (some conventions differ)
- Field is the clock in unitary gauge but the “ticking” $d\phi$ in proper time involves lapse: $d\tau = N dt = N(dt/d\phi)d\phi = (N/\dot\phi)d\phi$
- If $\mathcal{L}_\phi$ is function of X, its ADM analog is a function of the lapse N
- Generalization: EFT of inflation: note that $g^{00} = -1/N^2$ so the EFT literature sometimes writes $\mathcal{L}(g^{00}, t)$ (Cheung et al 2008)

EFT of Inflation

- In unitary gauge ϕ is a function of the temporal coordinate $X = -\dot\phi^2/N^2$ so that

$$\mathcal{L}_\phi(X,\phi) \to \mathcal{L}_\phi(N,t)$$

- Prototype: canonical kinetic term

$$\mathcal{L}_\phi(X,\phi) = -X/2 - V(\phi) = \dot\phi^2/2N^2 - V$$

- Two equivalent ways of proceeding: stay in unitary gauge and write the action for the metric, time shift to restore gauge invariance $t \to t+\pi$, $\pi = \delta\phi/(d\phi/dt)$ and work in gauge where the (Stuckelberg field π) $\phi(\mathbf{x},t) = \phi(t) + \delta\phi(\mathbf{x},t)$ dynamics are explicit

Unitary Gauge Action

- Expand the action

$$S = \int d^4x \sqrt{-g}\mathcal{L} = \int d^4x N \sqrt{h}\mathcal{L}$$

in the ADM metric fluctuations around FRW: lapse N, shift N^i and spatial metric

$$h_{ij} = a^2 e^{2\mathcal{R}}(\gamma_{ij} + h_{+ij} + h_{\times ij})$$

Note that ADM applies beyond linear theory so this completely defines the scalar and tensor metric in unitary gauge and coincides with the linear metric fluctuations to leading order

Unitary Gauge Action

- Varying the action for background

  $$S = \int d^4x N\sqrt{h}(\mathcal{L}_{\rm EH} + \mathcal{L}_\phi) = \int d^4x a^3(-3H^2/N + N\mathcal{L}_\phi)$$

  with respect to N (around $N = 1$) and a give

  $$\mathcal{L}_\phi + \mathcal{L}_{\phi,N} + 3H^2 = 0$$
  $$\mathcal{L}_\phi - 2\epsilon_H H^2 + 3H^2 = 0$$

  where $\epsilon_H = -d\ln H/d\ln a$ is the slow roll parameter

- For the additional contributions, $\mathcal{L}_\phi$ defines a pressure and $-\mathcal{L}_{\phi,N} - \mathcal{L}_\phi$ defines the energy density so these are just the inflationary Friedmann equations

  $$3H^2 = \frac{\dot\phi^2}{2} + V, \qquad 2\epsilon_H H^2 = 3H^2 + \frac{\dot\phi^2}{2} - V = \dot\phi^2$$

Curvature Fluctuations

- Similarly the EOMs for the linear fluctuations $A = \delta N, B, \mathcal{R}, h_+, h_\times$. for the quadratic action
- Expand the action to quadratic order in the perturbations (linear vanishes by virtue of the background EOM) and vary with respect to ADM variables: e.g. lapse perturbation

$$S_2 = \int d^4x N a^3 (\mathcal{L}_{\rm EH} + \mathcal{L}_\phi) = \int d^4x (C_{NN}\delta N^2 + C_{\mathcal{R}N}\mathcal{R}\delta N + \ldots)$$

where the Taylor expansion coefficients C_{NN},$C_{\mathcal{R}N}$ are the EFT functions of unitary time

- Since action does not involve time derivatives of the lapse its variation gives a constraint as does the shift (sometimes called the Hamiltonian and momentum constraints)

$$2C_{NN}\delta N + C_{\mathcal{R}N}\mathcal{R} + \ldots = 0$$

Curvature Fluctuations

- Eliminate the constraints and this leaves $\mathcal{R}$ as the dynamical variable

$$S_2 = \int d^4x \frac{a^3 \epsilon_H}{c_s^2} (\dot{\mathcal{R}}^2 + \frac{c_s^2 k^2}{a^2} \mathcal{R}^2)$$

where and the scalar sound speed is

$$c_s^2 = -\frac{\mathcal{L}_{,N}}{\mathcal{L}_{,NN} + 2\mathcal{L}_{,N}}$$

- Define the canonically normalized scalar as $u = z\mathcal{R}$ with $z = a\sqrt{\epsilon_H}/c_s$ and absorb $adt = d\eta$
- Vary wrt u gives the Mukhanov-Sasaki equation

$$\frac{d^2u}{d\eta^2} + \left(c_s^2 k^2 - \frac{1}{z}\frac{d^2 z}{d\eta^2} \right) u = 0$$

Curvature Fluctuations

- No slow roll conditions on ϵ_H or $d\epsilon_H/d\ln a$ so the EOM applies to "non-adiabatic" cases where they vary suddenly or have transiently large amplitude
- Whereas a large ϵ_H would end inflation, a transiently large $d\epsilon_H/d\ln a$ still gives sufficient inflation and can generate features (generalized slow roll technique) as well as PBHs
- Continue to the next order: cubic action S_3 given bispectrum (primordial non-Gaussianity), enhanced at low c_s^2
- Construction can be generalized for Lagrangians involving not just the lapse N but also the other ADM variables
- Curvature ADM quantities, intrinsic and extrinsic curvature, lead to modified gravitational forces – also useful for characterizing dark energy beyond k-essence

ADM Framework for Dark Energy EFT

- Dark energy may also be viewed in an EFT language where the unknown physics of dark energy is allowed to also interact with the geometric quantities of gravity
- ADM allows parameterization of all possibilities consistent with broken temporal diffeomorphism and unbroken spatial diffeomorphisms that lead to a scalar-tensor theory of gravity including all of the Horndenski theory
- Straightforward check of ghost, gradient instability in linear theory
- Unifying description for "building blocks" of dark energy (Gleyzes, Langois, Vernizzi 2015) - many different notations for this parameterize the same space of EFT functions
- Our ADM framework most closely follows arXiv:2002.07967 with1703.03797 as the dictionary to the EFT of dark energy

ADM EFT

- Recall the ADM line element

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)$$

where the preferred slicing given by the gradient of the scalar

$$\phi_{,\mu} = -\sqrt{-X} n_\mu$$

- Recall that the geometry of ADM is related to $n_{\mu;\nu}$, e.g. extrinsic curvature and acceleration

$$K_{\mu\nu} = n_{\nu;\mu} + n_\mu a_\nu$$

so it is constructed from

$$\phi_{;\mu\nu} = \sqrt{-X}(-K_{\nu\mu} + n_\nu a_\mu + n_\mu a_\nu - \beta n_\mu n_\nu)$$

ADM EFT

- Here the new term introduces a dependence on the shift N^i and $\dot{N}$ and comes from the derivative hitting X

$$\beta = -\frac{1}{2} n^\mu (\ln X)_{,\mu} = -\frac{\ddot{\phi}}{N\dot{\phi}} + \frac{\dot{N} - N^i \partial_i N}{N^2},$$

  which tracks the change in the lapse along the normal vector in field units since $X = -\dot{\phi}^2/N^2$ where $\phi(t)$ rather than $\phi = t$ for generality

- So to construct a theory with second order field dynamics we want to build the most general Lagrangian out of these ADM quantities
- For the acceleration, since we consider theories involving up to $\phi_{;\mu\nu}$, a^i and a_j are the only quantities that have one spatial sub/superscript. and must appear together

$$\alpha^i{}_j \equiv a^i a_j = h^{ik} (\ln N)_{,j} (\ln N)_{,k}, \quad \alpha \equiv \alpha^i{}_i.$$

ADM EFT Action

- ADM action becomes

$$S = \int d^4x N\sqrt{h}\mathcal{L}(N, K^i{}_j, R^i{}_j, \alpha^i{}_j, \beta; t)$$

  where the function is constructed to be spatially covariant, i.e. using the spatial metric and spatial covariant derivatives to form 3-scalars. Here R_{ij} is 3-Ricci tensor constructed from h_{ij} metric

- To make these considerations concrete, let's write down the Horndenski+ (aka DHOST) action for ϕ in terms of ADM

$$\begin{aligned}\mathcal{L} \quad &= \quad F_0 + F_1\Box\phi + F_2{}^{(4)}R + F_3{}^{(4)}G_{\mu\nu}\phi^{;\mu\nu} \\ &\quad + \sum_{i=1}^{5} A_i L_i^{(2)} + \sum_{i=1}^{10} B_i L_i^{(3)}\end{aligned}$$

  where $F_0, ...B_{10}$ are arbitrary functions of $(X, \phi) \rightarrow (N, t)$

Covariant ϕ Action

- Terms that are quadratic in second derivatives are

$$\begin{aligned}
L_1^{(2)} &= \phi_{;\mu\nu}\phi^{;\mu\nu} = \frac{\dot\phi^2}{N^2}(K^i{}_j K^j{}_i + \beta^2 - 2\alpha), \\
L_2^{(2)} &= (\Box\phi)^2 = \frac{\dot\phi^2}{N^2}(-K+\beta)^2, \\
L_3^{(2)} &= (\Box\phi)\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu} = \frac{\dot\phi^4}{N^4}\beta(K-\beta), \\
L_4^{(2)} &= \phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu\rho}\phi_{;\rho} = \frac{\dot\phi^4}{N^4}(\alpha-\beta^2), \\
L_5^{(2)} &= (\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu})^2 = \frac{\dot\phi^6}{N^6}\beta^2,
\end{aligned}$$

where traces are denoted as $K = K^i{}_i$, etc. (FRW curvature zero)

- Those that are cubic next....

Covariant ϕ Action

$$L_1^{(3)} = (\Box\phi)^3 = \frac{\dot\phi^3}{N^3}(-K+\beta)^3,$$

$$L_2^{(3)} = (\Box\phi)\phi_{;\mu\nu}\phi^{;\mu\nu} = \frac{\dot\phi^3}{N^3}(-K+\beta)(K^i{}_j K^j{}_i + \beta^2 - 2\alpha),$$

$$L_3^{(3)} = \phi_{;\mu\nu}\phi^{;\nu\rho}\phi^{;\mu}{}_{;\rho} = \frac{\dot\phi^3}{N^3}(-K^i{}_j K^j{}_k K^k{}_i + 3\alpha^i{}_j K^j{}_i + \beta^3 - 3\alpha\beta),$$

$$L_4^{(3)} = (\Box\phi)^2\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu} = -\frac{\dot\phi^5}{N^5}\beta(-K+\beta)^2,$$

$$L_5^{(3)} = (\Box\phi)\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu\rho}\phi_{;\rho} = \frac{\dot\phi^5}{N^5}(-K+\beta)(\alpha-\beta^2),$$

$$L_6^{(3)} = \phi_{;\mu\nu}\phi^{;\mu\nu}\phi^{;\rho}\phi_{;\rho\sigma}\phi^{;\sigma} = \frac{\dot\phi^5}{N^5}\beta(-K^i{}_j K^j{}_i - \beta^2 + 2\alpha),$$

$$L_7^{(3)} = \phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu\rho}\phi_{;\rho\sigma}\phi^{;\sigma} = \frac{\dot\phi^5}{N^5}(-\alpha^i{}_j K^j{}_i - \beta^3 + 2\alpha\beta),$$

$$L_8^{(3)} = \phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu\rho}\phi_{;\rho}\phi^{;\sigma}\phi_{;\sigma\xi}\phi^{;\xi} = \frac{\dot\phi^7}{N^7}\beta(\beta^2-\alpha),$$

$$L_9^{(3)} = \Box\phi(\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu})^2 = \frac{\dot\phi^7}{N^7}\beta^2(-K+\beta),$$

$$L_{10}^{(3)} = (\phi^{;\mu}\phi_{;\mu\nu}\phi^{;\nu})^3 = -\frac{\dot\phi^9}{N^9}\beta^3,$$

- Messy in either form but general

ADM perturbations

- Expand the ADM action to quadratic order in ADM variables around the flat FRW background ("b")

$$\bar{N} = 1, \quad \bar{N}^i = 0, \quad \bar{h}_{ij} = a^2\delta_{ij},$$

Taylor coefficients are the EFT functions

$$\begin{aligned} L\Big|_{\mathrm{b}} &= \mathcal{C}, \\ \frac{\partial L}{\partial Y^i{}_j}\Big|_{\mathrm{b}} &= \mathcal{C}_Y \delta^j{}_i, \\ \frac{\partial^2 L}{\partial Y^i{}_j \partial Z^k{}_\ell}\Big|_{\mathrm{b}} &= \mathcal{C}_{YZ}\delta^j{}_i\delta^\ell{}_k + \frac{\tilde{\mathcal{C}}_{YZ}}{2}(\delta^\ell{}_i\delta^j{}_k + \delta_{ik}\delta^{j\ell}) \end{aligned}$$

where $Y, Z \in \{N, K, R, \alpha, \beta\}$

ADM Quadratic Action

- Expand and keep quadratic terms in $\delta X, \delta Y \in \delta N, \delta K, \delta R, \delta\alpha, \delta\beta$ and their temporal and spatial derivatives; conceptually

  $$N\sqrt{h}\mathcal{L}_2 = a^3 \sum_{X,Y} (\ldots)\delta X \delta Y$$

  where $(\ldots)$ are messy combinations of the Taylor coefficients

- Isolate the scalar sector, lapse perturbation δN, shift $\partial_i\psi$, trace ζ

  $$N = 1 + \delta N, \quad N_i = \partial_i \psi, \quad h_{ij} = a^2 e^{2\zeta}\delta_{ij}$$

- Note that the α and β parameters involve $\delta\dot{N}$ and $\delta N_{,i}$.

ADM Quadratic Action

- Evaluate the ADM perturbations using the scalar metric components

$$
\begin{aligned}
\delta\sqrt{h} &= 3a^3\zeta, \\
\delta K^i{}_j &= (\dot\zeta - H\delta N)\delta^i{}_j - \frac{1}{a^2}\delta^{ik}\partial_k\partial_j\psi, \\
\delta K &= 3(\dot\zeta - H\delta N) - \frac{\partial^2\psi}{a^2}, \\
\delta_1 R^i{}_j &= -\frac{1}{a^2}(\delta^i{}_j\partial^2\zeta + \delta^{ik}\partial_k\partial_j\zeta), \\
\delta_2 R &= -\frac{2}{a^2}[(\partial\zeta)^2 - 4\zeta\partial^2\zeta] \qquad (3)
\end{aligned}
$$

- Vary the action with respect to $\delta N, \psi, \zeta$ to find the constraints and equations of motion

Stability Analysis

- Test for Ostrogradsky ghost and gradient instability
- Take Fourier modes for the scalar variable $\mathbf{u} = (\delta N, \psi, \zeta) \propto e^{(i\omega t + \mathbf{k}\cdot\mathbf{x})}$
- Write the quadratic action as

$$a^3 N\sqrt{h}\mathcal{L}_2 = \frac{1}{2}\mathbf{u}^\dagger \mathbf{K}\mathbf{u}$$

  whose variation with respect to $\mathbf{u}$ gives the EOM $\mathbf{K}\mathbf{u} = 0$

- Find conditions on the EFT coefficients that enforce allow a single dispersion relation $\omega^2 = c_s^2 k^2$ with a positive $c_s^2 \geq 0$ (or real sound speed)

Stability Analysis

- In general, supplying a quadratic term $C_{\beta\beta}$ in $\delta\dot{N}$ would lead to a second degree of freedom and solutions of the form $\omega^4 + ... = 0$, the hallmark of the Ostrogradsky ghost but there are specific conditions which kill the ω^4 term called Degenerate Higher Order Theories (DHOST)
- Remaining theories are in the Horndeski class, $F_{0...3}$ and the combination of the others that eliminates β^2 terms
- Within this class check for gradient stability $c_s^2 \geq 0$
- See arXiv:2002.07967 for the completion of analysis and explicit equations

Supplementary EFT Material

Stuckelberg Restoration

- $P(X, \phi)$ also provides an illustration of how the scalar is reintroduced by the so called "Stuckelberg" trick as a field that restores gauge invariance (temporal diffeomorphisms)
- Alternate to unitary gauge is to instead transform to spatially flat gauge where the ADM metric has no dynamics.
- Clarifies the origin of the ϵ_H difference in the scalar action
- EFT of inflation originally formulated in this language and with $g^{00} = -1/N^2$ so we will switch notion below.
- This language is also convenient for showing how the cubic action or non-Gaussianity of inflation

Effective Field Theory

- Now consider that $g^{00}+1$ is a small metric perturbation. A general function of the lapse may be expanded around this value in a Taylor series

$$S = \int d^4x\sqrt{-g}\Big[\frac{1}{2}M_{\rm Pl}^2 R + \sum_{n=0}^{\infty}\frac{1}{n!}M_n^4(t_u)(g_u^{00}+1)^n\Big],$$

- Varying action with respect to $g^{\mu\nu}$ we get the Einstein equations
- Constant term gives a cosmological constant whereas the $n=1$ term gives the effective stress tensor of the field in the background

$$\begin{aligned} H^2 &= -\frac{1}{3M_{\rm Pl}^2}\left[M_0^4 + 2M_1^4\right] \\ \dot H + H^2 &= -\frac{1}{3M_{\rm Pl}^2}\left[M_0^4 - M_1^4\right] \end{aligned}$$

where recall $\dot H = -H^2\epsilon_H$

Effective Field Theory

- Friedmann equation can thus eliminate $n = 0, 1$

$$\begin{aligned} M_0^4 &= -(3 - 2\epsilon_H) H^2 M_{\rm Pl}^2 \\ M_1^4 &= -H^2 \epsilon_H M_{\rm Pl}^2 \end{aligned}$$

- Now we an restore time slicing invariance or temporal diffs allowing for a general change in the time coordinate

$$t_u = t + \pi(t, x^i)$$

- In particle physics language this is the Stuckelberg trick and π is a Stuckelberg field.
- Transformation to arbitrary slicing is given by

$$g_u^{00} = \frac{\partial t_u}{\partial x^\mu} \frac{\partial t_u}{\partial x^\nu} g^{\mu\nu}$$

- Each $M_n^4(t_u = t + \pi)$ and hence carry extra Taylor expansion terms

Effective Field Theory

- In general, transformation mixes π and metric fluctuations $\delta g^{\mu\nu}$ including terms like

  $$\dot\pi\delta g^{00}, \quad \delta g\dot\pi, \quad \partial_i\pi g^{0i}, \quad \partial_i\pi\partial_j\pi\delta g^{ij}$$

  in the canonical linear theory calculation, the first three were the $\dot A$, $\dot H_L$, kB terms after integration by parts and the last is cubic order

- The lapse and shift are non-dynamical for the class of EFT we consider including $P(X,\phi)$, (beyond) Horndeski, so the most useful transformation to consider is to spatially flat gauge to eliminate dynamics in the spatial metric

- For this case, the gauge transformation of the curvature fluctuation tells us $\pi = -\mathcal{R}/H$

- We thus expect to recover the action for $\mathcal{R}$ from the action for π

Effective Field Theory

- In fact on scales below the horizon in most gauges the field fluctuations reduce to spatially flat gauge since curvature effects are negligible
- Spatially flat gauge extends the domain of validity extends even through the horizon if we neglect slow roll corrections
- In this case we can ignore the terms associated with the spatial pieces of the metric and replace

$$g_u^{00} = -(1+\dot{\pi})^2 + \frac{(\partial_i \pi)^2}{a^2}$$

- Each $g_u^{00} + 1$ factor carries terms that are linear and quadratic in π

$$(g_u^{00} + 1)^n = (-\dot{\pi})^n \sum_{i=0}^{n} \frac{2^{n-i} n!}{i!(n-i)!} \Pi^i$$

Effective Field Theory

- So each M_n^4 term contributes from π^n to π^{2n}

$$\Pi = \dot\pi \left(1 - \frac{(\partial_i \pi)^2}{a^2 \dot\pi^2}\right)$$

- For example M_2

$$\begin{aligned}(g_u^{00}+1)^2 &= \dot\pi^2 \left[4 + 4\dot\pi \left(1 - \frac{(\partial_i \pi)^2}{a^2 \dot\pi^2}\right) + \dot\pi^2 \left(1 - \frac{(\partial_i \pi)^2}{a^2 \dot\pi^2}\right)^2\right] \\ &= 4(\dot\pi^2 + \dot\pi^3 - \dot\pi \frac{(\partial_i \pi)^2}{a^2}) + \ldots\end{aligned}$$

implies both a cubic and quartic Lagrangian. To cubic order

$$S_\pi = \int d^4x \sqrt{-g} \Big[M_{\rm Pl}^2 \epsilon_H H^2 \left(\dot\pi^2 - \frac{(\partial_i \pi)^2}{a^2}\right) + 2M_2^4 (\dot\pi^2 + \dot\pi^3 - \dot\pi \frac{(\partial_i \pi)^2}{a^2}) + \ldots$$

Effective Field Theory

- Isolate the quadratic action

$$S_{\pi 2} = \int d^4x\sqrt{-g}\left[(M_{\rm Pl}^2 H^2 \epsilon_H + 2M_2^4)\dot\pi^2 + M_{\rm Pl}^2 \dot H \frac{(\partial_i \pi)^2}{a^2}\right]$$

and identify the sound speed from $\omega = (k/a)c_s$

$$c_s^{-2} = 1 + \frac{2M_2^4}{M_{\rm Pl}^2 \epsilon_H H^2}; \quad \Pi \sim \dot\pi\left(1 - \frac{1}{c_s^2}\right)$$

$$\begin{aligned} S_{\pi 2} &= \int dt d^3x (a^3 \epsilon_H H^2) M_{\rm Pl}^2 c_s^{-2}\left[\dot\pi^2 - c_s^2 \frac{(\partial_i\pi)^2}{a^2}\right] \\ &= \int d\eta d^3x \frac{z^2 H^2 M_{\rm Pl}^2}{2}\left[\left(\frac{\partial \pi}{\partial \eta}\right)^2 - c_s^2(\partial_i \pi)^2\right]\end{aligned}$$

where $z^2 = 2a^2\epsilon_H / c_s^2$

Effective Field Theory

- So a field redefinition canonically normalizes the field

  $$u = zH\pi M_{\rm Pl}$$

  brings the EFT action to canonical form (assuming M_n^4 =const.)

  $$\begin{aligned} S_u &= \int d\eta d^3x \left[\left(\frac{\partial u}{\partial \eta}\right)^2 - c_s^2(\partial_i u)^2 - 2u\frac{\partial u}{\partial \eta}\frac{d\ln z}{d\eta} + u^2\left(\frac{d\ln z}{d\eta}\right)^2 \right] \\ &= \int d\eta d^3x \left[\left(\frac{\partial u}{\partial \eta}\right)^2 - c_s^2(\partial_i u)^2 + \frac{u^2}{z}\frac{d^2 z}{d\eta} \right] \end{aligned}$$

  which is the generalization of the u field of canonical inflation

- Quantize this field, noting that $1/\sqrt{E}$ normalization factor goes to $1/\sqrt{kc_s}$ yielding the modefunction

  $$u = \frac{1}{\sqrt{2kc_s}}\left(1 - \frac{i}{kc_s\tilde\eta}\right)e^{-ikc_s\tilde\eta}$$

Effective Field Theory

- Curvature fluctuations then freezeout at $kc_s\tilde{\eta} = 1$ (sound horizon crossing) at a value

$$\mathcal{R} = -H\pi = \frac{c_s}{a\sqrt{2\epsilon_H}} \frac{1}{\sqrt{2kc_s}} \frac{i}{kc_s\tilde{\eta}M_{\rm Pl}} \approx \frac{-iH}{2k^{3/2}\sqrt{\epsilon_H c_s}M_{\rm Pl}}$$

- So

$$\Delta^2_{\mathcal{R}} = \frac{k^3|\mathcal{R}|^2}{2\pi^2} = \frac{H^2}{8\pi^2\epsilon_H c_s M^2_{\rm Pl}}$$

- Generalization is that the sound speed enters in two ways: boosts scalars over tensors by c_s and changes the epoch of freezeout between scalars and tensors

Effective Field Theory

- Returning to the original π action, since M_2^4 carries cubic term this requires a non-Gaussianity

$$S_\pi = \int d^4x\sqrt{-g}\left[-\frac{M_{\rm Pl}^2\dot H}{c_s^2}\left(\dot\pi^2 + c_s^2\frac{(\partial_i\pi)^2}{a^2}\right) + M_{\rm Pl}^2\dot H\left(1-\frac{1}{c_s^2}\right)\left(\dot\pi^3 - \dot\pi\frac{(\partial_i\pi)^2}{a^2}\right)\right] + \ldots$$

- For $c_s \ll 1$, spatial gradients dominate temporal derivatives

$$\partial_0 \to \omega, \partial_i \to k, \omega = kc_s/a$$

and leading order cubic term is $\dot\pi(\partial_i\pi)^2$

- Estimate the size of the non-Gaussianity by taking the ratio of cubic to quadratic at $c_s \ll 1$

$$\frac{\dot\pi(\partial_i\pi)^2}{a^2\dot\pi^2} \sim \frac{k\pi_{\rm rms}}{c_s a} \qquad \text{where} \quad \pi_{\rm rms} = \left(\frac{k^3|\pi|^2}{2\pi^2}\right)^{1/2}$$

Effective Field Theory

- Deep within the horizon $u = 1/\sqrt{2kc_s}$ and so

$$\begin{aligned}\frac{k\pi_{\rm rms}}{c_s a} &\sim \frac{k}{c_s a}\left(\frac{k^2}{2z^2H^2c_s M_{\rm Pl}^2}\right)^{1/2} \\ &\sim \left(\frac{kc_s}{aH}\right)^2 \frac{H}{M_{\rm Pl}\sqrt{\epsilon_H c_s}}\frac{1}{c_s^2} \\ &\sim \left(\frac{kc_s}{aH}\right)^2 \frac{\Delta_{\mathcal{R}}}{c_s^2} < 1\end{aligned}$$

- Since $kc_s/aH \sim \omega/H$ is a ratio of an energy scale to Hubble, the bound determines the strong coupling scale

$$\frac{\omega_{sc}}{H} \sim \frac{c_s}{\sqrt{\Delta_{\mathcal{R}}}} \sim 10^2 c_s$$

- For $c_s < 0.01$ the strong coupling scale is near the horizon and the effective theory has broken down before freezeout

Effective Field Theory

- Now consider a less extreme c_s
- Here the effective theory becomes valid at least several efolds before horizon crossing and we can make predictions within the theory
- Not surprisingly non-Gaussianity is enhanced by these self interactions and freezeout at $kc_s \sim aH$

$$\begin{aligned}\frac{k\pi_{\rm rms}}{c_s a} &\sim \frac{k}{c_s a}\left(\frac{1}{\epsilon_H c_s M_{\rm Pl}^2}\right)^{1/2} \\ &\sim \frac{kc_s}{aH}\frac{H}{\sqrt{\epsilon_H c_s} M_{\rm Pl}}\frac{1}{c_s^2} \\ &\sim \frac{\Delta_{\mathcal{R}}}{c_s^2}\end{aligned}$$

and so bispectrum is enhanced over the naive expectation by c_s^{-2}

Effective Field Theory

- More generally, each M_n^4 sets its own strong coupling scale

$$\frac{\mathcal{L}_n}{\mathcal{L}_2} \sim 1$$

These coincide if

$$\frac{M_n^4}{M_2^4} \sim \left(\frac{1}{c_s^2}\right)^{n-2}$$

which would be the natural prediction if the M_2 strong coupling scale indicated the scale of new physics and we take all allowed operators as order unity at that scale